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Magnons, their solitonic avatars and the Pohlmeyer reduction

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ABSTRACT: We study the solitons of the symmetric space sine-Gordon theories that arise once the Pohlmeyer reduction has been imposed on a sigma model with the symmetric space as target. Under this map the solitons arise as giant magnons that are relevant to string theory in the context of the AdS/CFT correspondence. In particular, we consider the cases S^n , $\mathbb{C}P^n$ and $SU(n)$ in some detail. We clarify the construction of the charges carried by the solitons and also address the possible Lagrangian formulations of the symmetric space sine-Gordon theories. We show that the dressing, or Bäcklund, transformation naturally produces solitons directly in both the sigma model and the symmetric space sine-Gordon equations without the need to explicitly map from one to the other. In particular, we obtain a new magnon solution in $\mathbb{C}P^3$. We show that the dressing method does not produce the more general “dyonic” solutions which involve non-trivial motion of the collective coordinates carried by the solitons.

KEYWORDS: Solitons Monopoles and Instantons, AdS-CFT Correspondence, Integrable Field Theories, Sigma Models

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1 Introduction

The AdS/CFT correspondence [1] is remarkable in so many ways. For example, there is an underlying integrable structure that allows one to interpolate from weak to strong coupling, and which enables many quantitative checks of the conjectured duality by exploring both sides of the correspondence (see [2] and the references therein). An example is provided by the “giant magnons” and the “dyonic giant magnons”, introduced by Hofman and Maldacena [3] and Dorey [4], respectively. They describe string configurations on curved space-times of the form $\mathbb{R}_t \times \mathfrak{M}$, with $\mathfrak{M} = F/G$ a symmetric space; for example, $S^n = \text{SO}(n+1)/\text{SO}(n)$. Then, the classical motion of the string is described by

a sigma model with target space \mathfrak{M} , and the Virasoro constraints, in a particular gauge, lead to the Pohlmeyer reduction of that sigma model [5, 6]. In turn, this gives rise to an associated integrable system that is a generalization of the sine-Gordon theory. These are the symmetric space sine-Gordon theories (SSSG), and giant magnons can be mapped into the soliton solutions to their equations-of-motion. Moreover, when the symmetric space is of indefinite signature, like $AdS_n = SO(2, n - 1)/SO(1, n - 1)$, similar ideas can be used to study also the motion of strings on curved spaces of the form $\mathfrak{M} \times S^1$, or even \mathfrak{M} . In the context of the AdS/CFT correspondence, giant magnons have been extensively used to study many aspects of superstrings in certain subspaces of $AdS_5 \times S^5$ [3, 7, 8] and $AdS_4 \times CP^3$ [9–11].

This work is a companion to [12], which provided a systematic study of the group theoretical interpretation of the Pohlmeyer reduction and the associated SSSG theories for symmetric spaces of definite, or indefinite, signature. The present work extends this to a discussion and construction of a class of soliton solutions using the dressing transformation method [13]. An important result that we establish is that the dressing method produces both the giant magnon *and* its soliton avatar in the SSSG theory at the same time, without the need to map one to other via the Pohlmeyer constraints. This is particularly useful because, in general, it is not easy to perform the map.

For cases including S^n the giant magnon solutions produced by the dressing method have been studied in [14] and they correspond to embeddings of the Hofman-Maldacena giant magnon [3] associated to $S^2 \subset S^5$. One major shortcoming of the dressing method is that, in the context of $S^n = SO(n + 1)/SO(n)$, it does not produce the Dorey’s dyonic giant magnon [4]. Nevertheless, this more general solution can be constructed by the dressing procedure by using the alternative formulation of S^3 as the symmetric space $SU(2) \times SU(2)/SU(2)$, which is isomorphic to the Lie group $SU(2)$. Embedding this solution back in the original formulation in terms of the symmetric space $S^n = SO(n + 1)/SO(n)$ shows that the dyonic solution involves non-trivial geodesic motion in the space of collective coordinates carried by the magnon/soliton. In this sense, the solutions have much in common with the dyonic generalization of the monopole in four-dimensional gauge theories coupled to an adjoint Higgs field (for example see [16]).

Other examples that we consider in this work are the complex projective spaces CP^n , which are realized as the symmetric spaces $SU(n + 1)/U(n)$. The case CP^3 is relevant to the AdS/CFT correspondence involving a spacetime $AdS_4 \times CP^3$ [17]. The known giant magnon solutions for this case [9–11, 18, 19] have all been obtained from the Hofman-Maldacena solution and the dyonic generalization of Dorey via embeddings of S^2 and S^3 in CP^n , respectively. Our results provide a class of new magnon/soliton solutions which cannot be obtained from embeddings of those for S^n . In addition, we show that there should exist an equivalent class of dyonic solutions in addition to the embeddings of Dorey’s dyon.

Finally, we consider the $SU(n)$ principal chiral models, which can be formulated as a symmetric space $SU(n) \times SU(n)/SU(n)$. For $n > 2$, these models admit several non-equivalent Pohlmeyer reductions and, therefore, they give rise to different SSSG theories. In this work we only consider the simplest cases, which correspond to the (parity invariant) homogeneous sine-Gordon theories [20]. The solitons of these theories have been studied

in [21] using a different formulation of the dressing transformation method based on representations of affine Lie algebras. Our results provide new expressions for them involving collective coordinates that clarify their composite nature in terms of basic $SU(2)$ solitons. More general reductions of the principal chiral models will be discussed elsewhere.

Notice that all the examples that we consider involve symmetric spaces of definite signature. In future work we will describe the generalization to symmetric spaces of indefinite signature relevant to discussing AdS_n , for example.

The plan of the paper is as follows. In section 2, we will formulate the sigma model with target space a symmetric space F/G in terms of a constrained F -valued field without introducing gauge fields. The relationship between this formulation and the approach used in [12] is summarized in appendix A. Using that formulation, in section 3 we will describe the Pohlmeyer reduction of the sigma model, and recover the formulation of the SSSG equations as zero-curvature conditions on a left-right asymmetric coset of the form $G/H_L^{(-)} \times H_R^{(+)}$ proposed in [12]. We will also address the possible Lagrangian formulations of these equations and clarify their symmetries and conserved quantities, which play an important rôle in the description of the soliton solutions. In section 4, we will review the already known giant magnons in the context of S^n and $\mathbb{C}P^n$, and we will discuss the relation between them and their relativistic SSSG solitonic avatars. In section 5, we will use the dressing transformation method to construct magnons and solitons following the approach of [22]. An important result of this section is that the dressing transformation is compatible with the Pohlmeyer reduction, and that this method provides directly both the magnon and its SSSG soliton without the need to map one into the other. In section 6, we will apply the method to the $SU(n)$ principal chiral model, and for $n = 2$ we will recover Dorey's dyonic giant magnon. In sections 7 and 8, we will apply the method to $\mathbb{C}P^n$ and S^n , respectively. In section 9, we discuss the possibility of finding solutions similar to Dorey's dyonic giant magnon by making the collective coordinates time dependent. Finally, section 10 contains our conclusions, and there are four appendices.

2 Symmetric space sigma model

Our story begins with a sigma model in $1+1$ dimensions whose target space is a symmetric space, that is a quotient of two Lie groups F/G equipped with an involution σ_- of F that fixes $G \subset F$:

$$\sigma_-(g) = g, \quad \forall g \in G. \tag{2.1}$$

Acting on \mathfrak{f} , the Lie algebra of F , the automorphism σ_- gives rise to the canonical orthogonal decomposition

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}, \quad \text{with} \quad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}, \tag{2.2}$$

where \mathfrak{g} and \mathfrak{p} are the $+1$ and -1 eigenspaces of σ_- , respectively, and \mathfrak{g} is the Lie algebra of G . Then, the sigma model with target space a symmetric space can be described as a sigma model with a field $f \in F$ where the G action $f \rightarrow fg^{-1}$, $g \in G$, is gauged. For instance, this is the approach described in the prequel [12] (see also [23]). However, for

present purposes, we find it more convenient to work directly in the coset F/G by defining the F -valued field

$$\mathcal{F} = \sigma_-(f)f^{-1} \tag{2.3}$$

and working directly with \mathcal{F} instead of f , which we can think of as an F -valued field subject to the constraint

$$\sigma_-(\mathcal{F}) = \mathcal{F}^{-1} . \tag{2.4}$$

In this formalism there is no need to introduce gauge fields and this simplification turns out to be useful. Of course, our approach in terms of involutions can easily be translated into the gauged sigma model language if need be (see appendix A).

We will also consider the principal chiral model which can either be considered as a symmetric space $G \times G/G$ as above, with σ_- being the involution that exchanges the two G factors, or we can simply take the target space to be $F = G$ itself, in which case the involution σ_- is not required. We will take the latter point of view in what follows.

The Lagrangian of the sigma model is simply

$$\mathcal{L} = -\frac{1}{8\kappa} \text{Tr} \mathcal{J}_\mu \mathcal{J}^\mu , \tag{2.5}$$

where

$$\mathcal{J}_\mu = \partial_\mu \mathcal{F} \mathcal{F}^{-1} . \tag{2.6}$$

Note that $\mathcal{F} \rightarrow \sigma_-(\mathcal{F}) = \mathcal{F}^{-1}$ is a symmetry of the action and equations-of-motion and therefore it is consistent to impose it by hand on the field \mathcal{F} from the start and, as mentioned above, in this formalism there are no gauge fields. The equations-of-motion for the group field are

$$\partial_\mu \mathcal{J}^\mu = 0 . \tag{2.7}$$

In other words, \mathcal{J}_μ provides the conserved currents corresponding to the global $F_L \times F_R$ symmetry of the sigma model with target space F (the principal chiral model) under which $\mathcal{F} \rightarrow U\mathcal{F}V$ for any $U, V \in F$. The left and right currents are

$$\mathcal{J}_\mu^L = \partial_\mu \mathcal{F} \mathcal{F}^{-1} = \mathcal{J}_\mu \quad \text{and} \quad \mathcal{J}_\mu^R = \mathcal{F}^{-1} \partial_\mu \mathcal{F} = \mathcal{F}^{-1} \mathcal{J}_\mu \mathcal{F} , \tag{2.8}$$

and we can define the corresponding conserved charges

$$\mathcal{Q}_L = \int_{-\infty}^{\infty} dx \partial_0 \mathcal{F} \mathcal{F}^{-1} \quad \text{and} \quad \mathcal{Q}_R = \int_{-\infty}^{\infty} dx \mathcal{F}^{-1} \partial_0 \mathcal{F} . \tag{2.9}$$

In the principal chiral model these charges are independent. However, in the F/G models the $F_L \times F_R$ symmetry is reduced by the constraint (2.4) so that they are invariant only under $\mathcal{F} \rightarrow \sigma_-(U)\mathcal{F}U^{-1}$ with $U \in F$. Taking (A.4) into account, these transformations correspond to $f \rightarrow Uf$, which specifies the global symmetries of the symmetric space sigma model in the gauged sigma model language. Consequently, in the F/G models the two charges are related by $\sigma_-(\mathcal{Q}_L) = -\mathcal{Q}_R$.

Since $\mathcal{J}_\pm = \partial_\pm \mathcal{F} \mathcal{F}^{-1}$, these currents trivially satisfy the Cartan-Maurer conditions¹

$$\partial_+ \mathcal{J}_- - \partial_- \mathcal{J}_+ - [\mathcal{J}_+, \mathcal{J}_-] = 0 . \tag{2.10}$$

¹ In our notation $x_+ = t + x$ and $x_- = t - x$

Then, the equations-of-motion (2.7), along with the identity (2.10), can be written in the form of a zero curvature condition:

$$\left[\partial_+ - \frac{\mathcal{J}_+}{1+\lambda}, \partial_- - \frac{\mathcal{J}_-}{1-\lambda} \right] = 0, \tag{2.11}$$

where λ is a spectral parameter. The residues at $\lambda = \pm 1$ then give the two equations

$$\mp \partial_{\pm} \mathcal{J}_{\mp} + \frac{1}{2} [\mathcal{J}_+, \mathcal{J}_-] = 0, \tag{2.12}$$

respectively, which are equivalent to (2.7) and (2.10).

3 The Pohlmeyer reduction

The Pohlmeyer reduction, at an algebraic level, involves imposing the conditions (see appendix A and [12])²

$$\partial_{\pm} \mathcal{F} \mathcal{F}^{-1} = f_{\pm} \Lambda_{\pm} f_{\pm}^{-1}, \tag{3.1}$$

where Λ_{\pm} are constant elements in a maximal abelian subspace \mathfrak{a} of \mathfrak{p} in (2.2) and $f_{\pm} \in F$. The natural degree-of-freedom left after the reduction is $\gamma = f_-^{-1} f_+$ which is valued in $G \subset F$. In order to see this, we act on (3.1) with σ_- . The left-hand sides become

$$\sigma_-(\partial_{\pm} \mathcal{F} \mathcal{F}^{-1}) = \partial_{\pm} \mathcal{F}^{-1} \mathcal{F} = -\mathcal{F}^{-1} \partial_{\pm} \mathcal{F} \tag{3.2}$$

while the right-hand sides transform into

$$\sigma_-(f_{\pm} \Lambda_{\pm} f_{\pm}^{-1}) = -\sigma_-(f_{\pm}) \Lambda_{\pm} \sigma_-(f_{\pm}^{-1}). \tag{3.3}$$

The two can be made consistent by requiring

$$\sigma_-(f_{\pm}) = \mathcal{F}^{-1} f_{\pm}, \tag{3.4}$$

and so

$$\sigma_-(\gamma) = f_-^{-1} \mathcal{F} \mathcal{F}^{-1} f_+ = \gamma, \tag{3.5}$$

which shows that $\gamma \in G$. Actually, it is clear that f_{\pm} are ambiguous since we could always right-multiply by the group of elements that commute with Λ_{\pm} , respectively. We shall soon see that this freedom leads to a gauge symmetry in the reduced model. Notice that once the reduction has been imposed the “left” F charges can be written

$$\mathcal{Q}_L = \int_{-\infty}^{\infty} dx (f_+ \Lambda_+ f_+^{-1} + f_- \Lambda_- f_-^{-1}). \tag{3.6}$$

Using (2.12) with $\mathcal{J}_{\pm} = f_{\pm} \Lambda_{\pm} f_{\pm}^{-1}$, we have

$$[f_+^{-1} \partial_- f_+ - \frac{1}{2} \gamma^{-1} \Lambda_- \gamma, \Lambda_+] = 0. \tag{3.7}$$

²In [12] the right-hand side had scales multipliers μ_{\pm} . In the present work, we will not indicate these factors. We can either re-introduce them by scaling x_{\pm} , or one can think of them as having been absorbed into Λ_{\pm} (see appendix A).

This implies that

$$f_+^{-1} \partial_- f_+ - \frac{1}{2} \gamma^{-1} \Lambda_- \gamma = A_-^{(R)} \quad (3.8)$$

where $A_-^{(R)}$ is an unknown element that satisfies $[A_-^{(R)}, \Lambda_+] = 0$ and, using (3.4), $\sigma_-(A_-^{(R)}) = A_-^{(R)}$. Therefore, $A_-^{(R)}$ takes values in \mathfrak{h}_+ , which is the Lie algebra of the subgroup $H^{(+)} \subset G$ of elements that commute with Λ_+ . Similarly, we have

$$\left[-f_+^{-1} \partial_+ f_+ + \gamma^{-1} \partial_+ \gamma + \frac{1}{2} \Lambda_+, \gamma^{-1} \Lambda_- \gamma \right] = 0, \quad (3.9)$$

which implies that

$$f_+^{-1} \partial_+ f_+ - \gamma^{-1} \partial_+ \gamma - \frac{1}{2} \Lambda_+ = \gamma^{-1} A_+^{(L)} \gamma. \quad (3.10)$$

Here, $[A_+^{(L)}, \Lambda_-] = 0$ and, using (3.4) once more, $\sigma_-(A_+^{(L)}) = A_+^{(L)}$. This shows that $A_+^{(L)} \in \mathfrak{h}_-$, which is the Lie algebra of the subgroup $H^{(-)} \subset G$ of elements that commute with Λ_- .

On the other hand, the integrability condition for (3.1) implies

$$[\partial_+ - f_+ \Lambda_+ f_+^{-1}, \partial_- - f_- \Lambda_- f_-^{-1}] = 0, \quad (3.11)$$

from which we deduce

$$[\partial_+ + f_+^{-1} \partial_+ f_+ - \Lambda_+, \partial_- + f_-^{-1} \partial_- f_- - \gamma^{-1} \Lambda_- \gamma] = 0. \quad (3.12)$$

Using (3.8) and (3.10), it gives

$$\left[\partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(L)} \gamma - \frac{1}{2} \Lambda_+, \partial_- + A_-^{(R)} - \frac{1}{2} \gamma^{-1} \Lambda_- \gamma \right] = 0, \quad (3.13)$$

which are the zero-curvature form of the Symmetric Space sine-Gordon (SSSG) equations-of-motion. Notice that, as a consequence of (3.1), this set of equations has a natural $H_L^{(-)} \times H_R^{(+)}$ gauge symmetry under which

$$f_{\pm} \longrightarrow f_{\pm} h_{\pm}^{-1}, \quad (3.14)$$

where h_{\pm} are local group elements in the subgroups $H^{(\pm)} \subset G$. Under this symmetry

$$\gamma \longrightarrow h_- \gamma h_+^{-1} \quad (3.15)$$

and

$$A_-^{(R)} \longrightarrow h_+ (A_-^{(R)} + \partial_-) h_+^{-1}, \quad A_+^{(L)} \longrightarrow h_- (A_+^{(L)} + \partial_+) h_-^{-1}. \quad (3.16)$$

This is exactly the result of [12].

The SSSG equations (3.13) are integrable and lead to an infinite set of conserved quantities which, as discussed in appendix B, include charges corresponding to the global part of the gauge group, and the energy and momentum. Since the conserved charges

play an important rôle, we will describe their construction in some detail. First of all, by projecting (3.13) onto \mathfrak{h}_+ and $\gamma \cdots \gamma^{-1}$ onto \mathfrak{h}_- yields the zero curvature conditions

$$[\partial_+ + A_+^{(R/L)}, \partial_- + A_-^{(R/L)}] = 0, \quad (3.17)$$

where we have defined the “missing” components of the gauge connections,

$$\begin{aligned} A_+^{(R)} &= \mathbf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(L)}\gamma), \\ A_-^{(L)} &= \mathbf{P}_{\mathfrak{h}_-}(-\partial_-\gamma\gamma^{-1} + \gamma A_-^{(R)}\gamma^{-1}). \end{aligned} \quad (3.18)$$

Eq. (3.17) gives rise to the conserved quantities associated to the global version of the $H_L^{(-)} \times H_R^{(+)}$ gauge transformations. Moreover, it enables the gauge fixing conditions that relate the SSSG equations to the non-abelian affine Toda equations ($A_{\pm}^{(R/L)} = A_{\mp}^{(R/L)} = 0$), and the gauge fixing conditions required for their Lagrangian formulation (see (3.34)). Then, it is important to notice that (3.17) holds provided that Λ_{\pm} give rise to the orthogonal decompositions

$$\mathfrak{f} = \text{Ker}(\text{Ad}_{\Lambda_{\pm}}) \oplus \text{Im}(\text{Ad}_{\Lambda_{\pm}}) \quad (3.19)$$

and, consequently, that

$$\begin{aligned} [\text{Ker}(\text{Ad}_{\Lambda_{\pm}}), \text{Ker}(\text{Ad}_{\Lambda_{\pm}})] &\subset \text{Ker}(\text{Ad}_{\Lambda_{\pm}}), \\ [\text{Ker}(\text{Ad}_{\Lambda_{\pm}}), \text{Im}(\text{Ad}_{\Lambda_{\pm}})] &\subset \text{Im}(\text{Ad}_{\Lambda_{\pm}}). \end{aligned} \quad (3.20)$$

This is always true if the symmetric space F/G is of definite signature (G compact), which is the only case considered in this work. An example where the decomposition (3.19) is not satisfied is provided by the “lightlike” Pohlmeyer reduction of the sigma model with target space AdS_n discussed in [12].

Under the gauge transformations (3.15)–(3.16),

$$A_{\pm}^{(R)} \longrightarrow h_+(A_{\pm}^{(R)} + \partial_{\pm})h_+^{-1}, \quad A_{\pm}^{(L)} \longrightarrow h_-(A_{\pm}^{(L)} + \partial_{\pm})h_-^{-1}. \quad (3.21)$$

Then, in order to construct gauge invariant conserved quantities, we will transform (3.17) into gauge invariant equations. First of all, we choose a gauge slice γ_0 such that any field γ can be written as

$$\gamma = \phi_L \gamma_0 \phi_R^{-1}, \quad (3.22)$$

with $\phi_L \in H^{(-)}$ and $\phi_R \in H^{(+)}$. Under gauge transformations, γ_0 remains invariant while $\phi_L \rightarrow h_- \phi_L$ and $\phi_R \rightarrow h_+ \phi_R$. Then, it can be easily checked that

$$\tilde{A}_{\pm}^{(R)} = \phi_R^{-1}(A_{\pm}^{(R)} + \partial_{\pm})\phi_R \quad \text{and} \quad \tilde{A}_{\pm}^{(L)} = \phi_L^{-1}(A_{\pm}^{(L)} + \partial_{\pm})\phi_L \quad (3.23)$$

are gauge invariant and, moreover, that

$$[\partial_+ + \tilde{A}_+^{(R/L)}, \partial_- + \tilde{A}_-^{(R/L)}] = 0. \quad (3.24)$$

In general, $H^{(\pm)}$ will be of the form $U(1)^{p_{\pm}} \times H_{\text{ss}}^{(\pm)}$, where p_{\pm} are positive integers and $H_{\text{ss}}^{(\pm)}$ are semi-simple factors. This allows one to write

$$\phi_{R/L} = e^{\alpha_{R/L}} \varphi_{R/L}, \quad (3.25)$$

where $e^{\alpha_{R/L}} \in U(1)^{p\pm}$ and $\varphi_{R/L} \in H_{ss}^{(\pm)}$. Then, the projection of (3.24) on the Lie algebras of $H_{ss}^{(\pm)}$ and $U(1)^{p\pm}$ provide two different types of gauge invariant conserved quantities. Namely, the projection of (3.24) on the Lie algebra of $U(1)^{p\pm}$ shows that the currents

$$J_{R/L}^\mu = \epsilon^{\mu\nu} \mathbf{P}_{u(1)^{p\pm}}(\tilde{A}_\nu^{(R/L)}) = \epsilon^{\mu\nu} \left(\mathbf{P}_{u(1)^{p\pm}}(A_\nu^{(R/L)}) + \partial_\nu \alpha_{R/L} \right) \quad (3.26)$$

are conserved. They lead to the “local” gauge invariant conserved quantities

$$Q_{R/L} = \int_{-\infty}^{+\infty} dx J_{R/L}^0 = \alpha_{R/L}(+\infty) - \alpha_{R/L}(-\infty) + \int_{-\infty}^{+\infty} dx \mathbf{P}_{u(1)^{p\pm}}(A_1^{(R/L)}) \quad (3.27)$$

which take values in the (abelian) Lie algebra of $U(1)^{p\pm}$. On the other hand, the projection of (3.24) on the Lie algebra of $H_{ss}^{(\pm)}$ provide the “non-local” conserved quantities given by the path ordered exponentials

$$\begin{aligned} \Omega_{R/L} &= \mathbf{P} \exp \left(- \int_{-\infty}^{\infty} dx \mathbf{P}_{\mathfrak{h}_{ss}^{(\pm)}}(\tilde{A}_1^{(R/L)}) \right) \\ &= \varphi_{R/L}^{-1}(+\infty) \mathbf{P} \exp \left(- \int_{-\infty}^{\infty} dx \mathbf{P}_{\mathfrak{h}_{ss}^{(\pm)}}(A_1^{(R/L)}) \right) \varphi_{R/L}(-\infty), \end{aligned} \quad (3.28)$$

which take values in $H_{ss}^{(\pm)}$. Notice that the conserved charges (3.27) and (3.28) are *not* the same as the conserved charges of the original sigma model Q_R and Q_L . In particular, the former are Lorentz invariant (see appendix B) while the latter are not. In certain circumstances, and in particular for the soliton solutions, it can transpire that for particular configurations $\mathbf{P}_{\mathfrak{h}_{ss}^{(\pm)}}(A_\pm^{(R/L)})$ take values in an abelian subalgebra of $\mathfrak{h}_{ss}^{(\pm)}$, and $\varphi_{R/L}$ in the corresponding abelian subgroup of $H_{ss}^{(\pm)}$ (for all x). In this case, the path ordering in (3.28) is unnecessary and we can write $\Omega_{R/L} = \exp Q_{R/L}^{(ss)}$ for abelian charges $Q_{R/L}^{(ss)}$ taking values in the relevant abelian subalgebras of $\mathfrak{h}_{ss}^{(\pm)}$.

It is worth remarking that ϕ_L and ϕ_R are subject to an ambiguity whenever a particular field configuration is invariant under a certain subgroup of $H_L^{(-)} \times H_R^{(+)}$. As an example, consider the vacuum configuration itself, $\gamma = 1$ with $A_\pm^{(R/L)} = 0$. It is invariant under the global vector subgroup of $H_L^{(-)} \times H_R^{(+)}$, which means that ϕ_L and ϕ_R are uniquely defined only modulo $\phi_L \rightarrow \phi_L U$ and $\phi_R \rightarrow \phi_R U$, with $U \in H_L^{(-)} \cap H_R^{(+)}$. Consequently, the local charges carried by the vacuum solution are unambiguously defined only up to $Q_L \sim Q_L + \rho$ and $Q_R \sim Q_R + \rho$, for $\rho \in u(1)^{p-} \cap u(1)^{p+}$. So, in a sense, only the combination $Q_L - Q_R$ is an unambiguously well-defined charge. The significance of this and its relation to spontaneous symmetry breaking will become clearer when we discuss the Lagrangian formulation of the SSSG equations later in this section.

The energy-momentum tensor is constructed in appendix (B), and leads to the following expression for the energy of a configuration

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \int dx \text{Tr} \left[- (\partial_+ \gamma \gamma^{-1} + A_+^{(L)})^2 + A_+^{(R)2} \right. \\ &\quad \left. - (\gamma^{-1} \partial_- \gamma - A_-^{(R)})^2 + A_-^{(L)2} + \Lambda_+ \gamma^{-1} \Lambda_- \gamma - \Lambda_+ \Lambda_- \right], \end{aligned} \quad (3.29)$$

relative to $\mathcal{E} = 0$ for $\gamma = 1$. We will find that the dressing procedure always produces soliton solutions of (3.13) which have vanishing gauge fields $A_-^{(R)} = A_+^{(L)} = 0$ and which satisfy the conditions

$$\mathbf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma) = 0, \quad \mathbf{P}_{\mathfrak{h}_-}(\partial_-\gamma\gamma^{-1}) = 0, \quad (3.30)$$

and hence $A_\mu^{(R)} = A_\mu^{(L)} = 0$. Then, the conserved charges only get contributions from the boundary terms $\phi_{R/L}(\pm\infty)$ and, within the examples discussed in the following sections, these turn out to be non-trivial only in the principal chiral models (see section 6). As a consequence only the principal chiral model solitons are charged under the SSSG $H_L^{(-)} \times H_R^{(+)}$ symmetry. In contrast, the solitons do always carry sigma model charge $\mathcal{Q}_{L,R}$. In section 9, we will see how to produce solitons in the reduced symmetric space sigma models which carry non-trivial $H_L^{(-)} \times H_R^{(+)}$ charges; however, one needs to go beyond the dressing transformation to produce them.

Lagrangian formulations. It is only natural to search for a relativistically invariant Lagrangian formulation of the SSSG equations (3.13). However, as we shall see and as has been pointed out elsewhere [12, 24, 25] there are problems that arise in pursuing this idea, and it may be that the SSSG equations themselves should be used as a basis for a canonical quantization without recourse to a Lagrangian.

Lagrangian formulations are only known when $H^{(-)}$ and $H^{(+)}$ are isomorphic and of the form [12, 24]

$$H_R^{(+)} = \epsilon_R(H), \quad H_L^{(-)} = \epsilon_L(H), \quad (3.31)$$

where H is a Lie group and $\epsilon_{L,R}: H \rightarrow G$ are two ‘‘anomaly-free’’ group homomorphisms that descend to embeddings of the corresponding Lie algebras \mathfrak{h} and \mathfrak{g} .³ Then, each non-equivalent choice of ϵ_L and ϵ_R gives rise to a different Lagrangian formulation. This is obtained by writing

$$A_+^{(L)} = \epsilon_L(\mathcal{A}_+), \quad A_-^{(R)} = \epsilon_R(\mathcal{A}_-), \quad (3.32)$$

where \mathcal{A}_\pm take values in \mathfrak{h} , and imposing the constraints

$$\begin{aligned} \mathbf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma + \gamma^{-1}\epsilon_L(\mathcal{A}_+)\gamma) &= \epsilon_R(\mathcal{A}_+), \\ \mathbf{P}_{\mathfrak{h}_-}(-\partial_-\gamma\gamma^{-1} + \gamma\epsilon_R(\mathcal{A}_-)\gamma^{-1}) &= \epsilon_L(\mathcal{A}_-), \end{aligned} \quad (3.33)$$

which can be viewed as a set of partial gauge fixing conditions [12, 25].⁴ They can be written as

$$A_-^{(L)} = \epsilon_L(\mathcal{A}_-), \quad A_+^{(R)} = \epsilon_R(\mathcal{A}_+), \quad (3.34)$$

where $A_-^{(L)}$ and $A_+^{(R)}$ are the ‘‘missing’’ components defined in (3.18). These conditions reduce the $H_L^{(-)} \times H_R^{(+)}$ gauge symmetry (3.15) to

$$\gamma \longrightarrow \epsilon_L(h)\gamma\epsilon_R(h^{-1}), \quad h \in H, \quad (3.35)$$

³Here, anomaly free simply means that $\text{Tr}(\epsilon_L(a)\epsilon_L(b)) = \text{Tr}(\epsilon_R(a)\epsilon_R(b))$ for all $a, b \in \mathfrak{h}$.

⁴In [12], it was shown that this interpretation is consistent provided that the orthogonal decompositions (3.19) hold, which is always true if the symmetric space is of definite signature.

under which \mathcal{A}_μ transforms as a gauge connection:

$$\mathcal{A}_\mu \longrightarrow h(\mathcal{A}_\mu + \partial_\mu)h^{-1}. \quad (3.36)$$

In addition, the gauge conditions (3.33) leave a residual symmetry under the global (abelian) transformations

$$\gamma \longrightarrow e^{\epsilon_L(\rho)}\gamma e^{+\epsilon_R(\rho)}, \quad \mathcal{A}_\mu \longrightarrow \mathcal{A}_\mu, \quad (3.37)$$

where e^ρ is in the centre of H .

The gauge-fixed equations-of-motion are then

$$[\partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}\epsilon_L(\mathcal{A}_+)\gamma, \partial_- + \epsilon_R(\mathcal{A}_-)] = \frac{1}{4}[\Lambda_+, \gamma^{-1}\Lambda_-\gamma] \quad (3.38)$$

and these follow as the equations-of-motion of the Lagrangian density

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{WZW}(\gamma) + \frac{1}{2\pi}\text{Tr} & \left(-\epsilon_L(\mathcal{A}_+)\partial_-\gamma\gamma^{-1} + \epsilon_R(\mathcal{A}_-)\gamma^{-1}\partial_+\gamma \right. \\ & \left. + \gamma^{-1}\epsilon_L(\mathcal{A}_+)\gamma\epsilon_R(\mathcal{A}_-) - \epsilon_L(\mathcal{A}_+)\epsilon_L(\mathcal{A}_-) - \frac{1}{4}\Lambda_+\gamma^{-1}\Lambda_-\gamma \right), \end{aligned} \quad (3.39)$$

where $\mathcal{L}_{WZW}(\gamma)$ is the usual WZW Lagrangian density for γ . In fact this theory is the asymmetrically gauged WZW model for G/H specified by $\epsilon_{R/L}$ with a potential. Notice that the partial gauge-fixing constraints (3.33) now appear as the equations-of-motion of the gauge connection. If we take the Lagrangian (3.40) as the basis for a QFT then many questions arise. For instance are the resulting QFTs independent of the choice of the form of the gauge group; i.e., independent of ϵ_L and ϵ_R ? In many cases, it can be shown that different theories are actually related by a target space T-duality symmetry [26], hinting that they are equivalent at the quantum level.

Now we turn to the symmetries of the Lagrangian theory and the relation with the conserved charges Q_L and Q_R of the SSSG equations. Since our primary interest is in the soliton solutions, it is a fact that the transformations $\phi_{R/L}$ that bring γ to the gauge slice (3.22) lie in an abelian subgroup of $H_L^{(-)}$ and $H_R^{(+)}$. As a consequence there are associated local conserved currents and charges.⁵ Then, for our purposes, it will be enough to restrict the following discussion to the case of abelian H . Then, the Lagrangian (3.40) is symmetric under the (abelian) global transformations

$$\gamma \longrightarrow e^{\epsilon_L(u)}\gamma e^{-\epsilon_R(v)}, \quad \mathcal{A}_\mu \longrightarrow \mathcal{A}_\mu, \quad (3.40)$$

where u, v take values in \mathfrak{h} . For $u = v$ this is just a global gauge transformation of the form (3.35) while for $u = -v$ it is a global symmetry transformation of the form (3.37). Following standard means, we can derive the corresponding Noether currents as follows (for instance, see [27]). Consider the variation of the Lagrangian action $S = \int d^2x \mathcal{L}$ under an infinitesimal transformation of the form

$$\gamma^{-1}\delta\gamma = \gamma^{-1}\epsilon_L(u)\gamma - \epsilon_R(v), \quad \delta\mathcal{A}_\mu = 0, \quad (3.41)$$

⁵For a more general configuration, we would have to separate out the abelian factors in $H = \text{U}(1)^p \times H_{\text{ss}}$ in an obvious way, as we did in the last section, and describe the semi-simple part in terms of non-local conserved charges. However, for the soliton solutions this technology is unnecessary.

with $u = u(t, x)$ and $v = v(t, x)$. It reads

$$\begin{aligned}
 \delta S &= \int d^2x \operatorname{Tr} \left(\left[\partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} \epsilon_L(\mathcal{A}_+) \gamma - \frac{1}{2} \Lambda_+, \partial_- + \epsilon_R(\mathcal{A}_-) - \frac{1}{2} \gamma^{-1} \Lambda_- \gamma \right] \gamma^{-1} \delta \gamma \right) \\
 &= \int d^2x \operatorname{Tr} \left((\partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+) (u - v) + \right. \\
 &\quad \left. + \partial_- \left(\mathbf{P}_{\mathfrak{h}_+} (\gamma^{-1} \partial_+ \gamma + \gamma^{-1} \epsilon_L(\mathcal{A}_+) \gamma) - \epsilon_R(\mathcal{A}_+) \right) \epsilon_R(v) + \right. \\
 &\quad \left. + \partial_+ \left(\mathbf{P}_{\mathfrak{h}_-} (-\partial_- \gamma \gamma^{-1} + \gamma \epsilon_R(\mathcal{A}_-) \gamma^{-1}) - \epsilon_L(\mathcal{A}_-) \right) \epsilon_L(u) \right)
 \end{aligned} \tag{3.42}$$

Then, the condition that δS vanishes for any u, v provides the conservation equations we are looking for. Using the constraints (3.33), they read

$$\partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ = 0, \tag{3.43}$$

which are the conservation equations of the current

$$J^\mu = \epsilon^{\mu\nu} \mathcal{A}_\nu. \tag{3.44}$$

Notice that, since $\delta S = 0$ for $u = v$, J^μ is the Noether current associated to the abelian global transformations (3.37), and there is no conserved current associated to global gauge transformations.

J^μ is clearly not invariant under the gauge transformations (3.35)–(3.36), which in this (abelian) case are of the form

$$\gamma \longrightarrow e^{\epsilon_L(u)} \gamma e^{-\epsilon_R(u)}, \quad \mathcal{A}_\mu \longrightarrow \mathcal{A}_\mu - \partial_\mu u. \tag{3.45}$$

In order to construct gauge invariant conserved quantities, we write the SSSG gauge slice (3.22) as

$$\gamma = \phi_L \gamma_0 \phi_R^{-1} = e^{\epsilon_L(\alpha+\beta)} \gamma_0 e^{-\epsilon_R(\alpha-\beta)} \tag{3.46}$$

such that, under (3.45), $\alpha \rightarrow \alpha + u$ while β and γ_0 remain fixed. Then, the gauge invariant Noether current associated to the abelian global transformations (3.37) is

$$\tilde{J}^\mu = \epsilon^{\mu\nu} (\mathcal{A}_\nu + \partial_\nu \alpha), \tag{3.47}$$

which provides the Noether charge

$$Q^N = \alpha(+\infty) - \alpha(-\infty) + \int_{-\infty}^{+\infty} dx \mathcal{A}_1. \tag{3.48}$$

Similarly to the case of $Q_{R/L}$ discussed in the previous section, the definition of Q^N is subject to an ambiguity whose form can be found by looking at the vacuum configuration $\gamma^{\text{vac}} = 1$. Namely, since it is invariant under $\gamma \rightarrow e^\rho \gamma e^{-\rho}$, the field α in (3.46) is only

defined up to $\alpha \rightarrow \alpha + \eta$ for any field $\eta \in \mathfrak{h}$ such that $\epsilon_L(\eta) = \epsilon_R(\eta)$. Consequently, the Noether charge is defined only modulo

$$Q^N \longrightarrow Q^N + q \quad \text{for each } q \in \mathfrak{h} \quad \text{such that } (\epsilon_L - \epsilon_R)(q) = 0. \quad (3.49)$$

In the Lagrangian formulation, this ambiguity has a physical interpretation. Notice that each constant $\rho \in \mathfrak{h}$ such that $(\epsilon_L - \epsilon_R)(\rho) = 0$ generates a symmetry transformation of the form (3.37) that changes $\gamma^{\text{vac}} = 1$; namely, $1 \rightarrow e^{2\epsilon_L(\rho)}$. Then, the ambiguity reflects the impossibility of defining a Noether charge for global symmetry transformations that do not leave the vacuum configuration invariant.

The relationship between the SSSG conserved quantities $Q_{R/L}$ (or $Q_{R/L}^{(\text{ss})}$) and Q^N can be easily derived by taking into account (3.27), (3.28), (3.32) and, according to (3.46), $\alpha_{R/L} = \epsilon_{R/L}(\alpha \mp \beta)$. It reads

$$Q_L = \epsilon_L(Q^N + Q^T), \quad Q_R = \epsilon_R(Q^N - Q^T), \quad (3.50)$$

where

$$Q^T = \beta(+\infty) - \beta(-\infty) \quad (3.51)$$

is a kind of (gauge invariant) topological, or kink, charge. The definition of Q^T only makes sense if the global symmetry (3.37), which corresponds to $\beta \rightarrow \beta + \rho$ in (3.46), changes the vacuum configuration and gives rise to non-trivial boundary conditions for γ . Consequently, the definition of Q^T is also subject to an ambiguity whose form can be found by looking again at $\gamma^{\text{vac}} = 1$. Since it is invariant under $\gamma \rightarrow e^\rho \gamma e^{-\rho}$, the field β in (3.46) is defined only up to $\beta \rightarrow \beta + \eta$ for any $\eta \in \mathfrak{h}$ such that $\epsilon_L(\eta) = -\epsilon_R(\eta)$, which means that the topological charge is defined modulo

$$Q^T \longrightarrow Q^T + q \quad \text{for each } q \in \mathfrak{h} \quad \text{such that } (\epsilon_L + \epsilon_R)(q) = 0. \quad (3.52)$$

To summarize, in the Lagrangian formulation the soliton configurations are expected to carry both Noether Q^N and topological Q^T charges. It is worth noticing that the combination of the SSSG charges that is free of ambiguities reads

$$Q_L - Q_R = (\epsilon_L - \epsilon_R)(Q^N) + (\epsilon_L + \epsilon_R)(Q^T) \quad (3.53)$$

which, not surprisingly, is also free of the ambiguities (3.49) and (3.52). Looking at this equation, it is worthwhile to recall that the different Lagrangian formulations of a set of SSSG equations are related by $H_L^{(-)} \times H_R^{(+)}$ gauge transformations, and that $Q_L - Q_R$ is gauge invariant and, hence, independent of the choice of $\epsilon_{R/L}$. Moreover, since T-duality transformations interchange Noether and topological charges, (3.53) is consistent with the expectation that the different Lagrangian theories are indeed related by T-duality symmetries.

The physical meaning of the charges and their ambiguities becomes clearer once we consider examples of particular gaugings. The most obvious kind of gauging that can always be chosen is

$$\epsilon_L(\alpha) = \epsilon_R(\alpha) = \alpha, \quad (3.54)$$

which corresponds to gauging the vector subgroup of $H_L^{(-)} \times H_R^{(+)}$. Then, the value of Q^N is meaningless, and the solitons are kinks characterized by the topological charge Q^T . In this case, the Noether current corresponds to the axial transformations $\gamma \rightarrow e^\rho \gamma e^\rho$, which do not leave the vacuum invariant. This means that at the classical level the symmetry is spontaneously broken. Of course at the quantum level this would have to be re-evaluated.

Since we are assuming that H is abelian, one can also gauge the axial vector subgroup by taking

$$\epsilon_L(\alpha) = -\epsilon_R(\alpha) = \alpha, \tag{3.55}$$

In this case, Q^N is free of ambiguities. It is the Noether charge corresponding to vector transformations $\gamma \rightarrow e^\beta \gamma e^{-\beta}$ that leave the vacuum invariant and, therefore, do not break the symmetry. In contrast, since the vacuum configuration is unique up to (axial) gauge transformations, the topological charge Q^T is arbitrary. Therefore, solitons are similar to Q-balls. Other choices of $\epsilon_{R/L}$ give rise to different interpretations of solitons as some sort of dyons that carry both Noether and topological charge.

As we have mentioned, the dressing procedure always produces soliton solutions of (3.13) which have vanishing gauge fields $A_-^{(R)} = A_+^{(L)} = 0$. This means that the soliton solutions are valid solutions of the gauged WZW model (3.38) and (3.33) for any choice of gauging with $\mathcal{A}_\mu = 0$. Consequently, as is clear from (3.48) and (3.51), the charges can be calculated in terms of boundary values of α and β or, equivalently, $\alpha_{R/L}$. The mass of the solitons can be calculated from the energy-momentum tensor of the gauged WZW theory which leads (up to an overall factor) to (3.29) with the identifications (3.32). Notice that, as a consequence of the anomaly free condition, $\text{Tr}(A_\pm^{(R/L)^2}) = \text{Tr}(A_\pm^{(L/R)^2}) = \text{Tr}(A_\pm^2)$.

3.1 $\mathbb{C}P^2$ example

In order to have an explicit example of the SSSG equations and their Lagrangian formulation consider $\mathfrak{M} = \mathbb{C}P^2$. Since the symmetric space $\mathbb{C}P^2 = \text{SU}(3)/\text{U}(2)$ has rank one, there is a unique Pohlmeyer reduction for which we can take (up to conjugation)

$$\Lambda_+ = \Lambda_- \equiv \Lambda = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.56}$$

In this case $H^{(\pm)} = \text{U}(1)$ and we can use both vector or axial gauging to achieve a Lagrangian formulation. We can parameterize the group element and gauge field as (see (3.22))

$$\gamma = e^{a_L h} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta e^{i\varphi} & \sin \theta \\ 0 & -\sin \theta & \cos \theta e^{-i\varphi} \end{pmatrix} e^{-a_R h}, \tag{3.57}$$

where $h = i \text{diag}(1, 1, -2)$ is the generator of \mathfrak{h} , and $\alpha_{R/L} = a_{R/L} h$.

If we choose vector gauging, then solving the conditions (3.33) for \mathcal{A}_μ and writing $\alpha_L + \alpha_R = 2\alpha$ like in (3.46), yields the gauge invariant Noether current for axial transformations:

$$\tilde{J}^{(V)\mu} = \epsilon^{\mu\nu} (\mathcal{A}_\nu + \partial_\nu \alpha) = \frac{1}{3} \left(\left(\frac{1}{2} + 2 \cot^2 \theta \right) \partial^\mu (a_L - a_R) + \cot^2 \theta \partial^\mu \varphi \right) h. \tag{3.58}$$

The two remaining equations, also depend only on $\alpha_L - \alpha_R$, as one expects since the combination $\alpha_L + \alpha_R$ has been gauged away, and so it is convenient to define $a_L - a_R = \psi/2$ (ψh corresponds to 4β in (3.46)):

$$\begin{aligned} \partial^\mu \partial_\mu \psi &= -4 \cos \theta \sin \varphi, \\ \partial^\mu \partial_\mu \theta + \frac{\cos \theta}{\sin^3 \theta} \partial_\mu (\varphi + \psi) \partial^\mu (\varphi + \psi) &= -\sin \theta \cos \varphi. \end{aligned} \tag{3.59}$$

These equations along with the continuity of the Noether current follow from the Lagrangian

$$\mathcal{L} = \partial_\mu \theta \partial^\mu \theta + \frac{1}{4} \partial_\mu \psi \partial^\mu \psi + \cot^2 \theta \partial_\mu (\psi + \varphi) \partial^\mu (\psi + \varphi) + 2 \cos \theta \cos \varphi. \tag{3.60}$$

The Lagrangian manifests the axial symmetry $\psi \rightarrow \psi + a$. In this case, the vacuum configuration is degenerate, $\gamma^{\text{vac}} = e^{h\psi/2}$, i.e. $\theta = \varphi = 0$ with $0 \leq \psi < 4\pi$, and the definition of the Noether charge does not make sense. Then, the solitons are characterized by the charge Q^T , which is simply the kink charge

$$Q^T = \frac{1}{4} [\psi(\infty) - \psi(-\infty)] h. \tag{3.61}$$

Classically the axial symmetry would be spontaneously broken. In the quantum theory, this would have to be re-evaluated since the theory is defined in 1 + 1-dimensional spacetime, Goldstone's Theorem does not apply and the would-be Goldstone modes should be strongly coupled giving rise to a mass gap. A related issue is the fact that the Lagrangian does not have a good expansion in terms of fields around their vacuum values due to the $\cot^2 \theta$ term in the Lagrangian.

On the other hand, if we choose axial gauging, then $\alpha_L - \alpha_R = 2\alpha$, and the gauge invariant conserved Noether current for vector transformations is

$$\tilde{J}^{(A)\mu} = \epsilon^{\mu\nu} (\mathcal{A}_\nu + \partial_\nu \alpha) = \frac{1}{1 + 4 \cot^2 \theta} \left(\frac{3}{2} \partial^\mu (a_L + a_R) - \cot^2 \theta \epsilon^{\mu\nu} \partial_\nu \varphi \right) h. \tag{3.62}$$

Then, we can define $a_L + a_R = \tilde{\psi}/2$ ($\tilde{\psi} h = 4\beta$ in (3.46)) and the corresponding Lagrangian is

$$\begin{aligned} \mathcal{L} = & \partial_\mu \theta \partial^\mu \theta + \frac{1}{1 + 4 \cot^2 \theta} \left(\frac{9}{4} \partial_\mu \tilde{\psi} \partial^\mu \tilde{\psi} + \cot^2 \theta \partial_\mu \varphi \partial^\mu \varphi - 6 \cot^2 \theta \epsilon^{\mu\nu} \partial_\mu \tilde{\psi} \partial_\nu \varphi \right) \\ & + 2 \cos \theta \cos \varphi, \end{aligned} \tag{3.63}$$

which manifests the vector symmetry $\tilde{\psi} \rightarrow \tilde{\psi} + a$. In this case the vacuum is non-degenerate, $\gamma^{\text{vac}} = 1$, because $\tilde{\psi}$ is not a good coordinate around $\theta = \varphi = 0$,⁶ and the vacuum is invariant under the (vector) symmetry. Consequently, the definition of Q^T does not make sense. In contrast,

$$Q^N = \int_{-\infty}^{+\infty} dx \tilde{J}_0^{(A)} \tag{3.64}$$

is unambiguously defined. Moreover, in this case the Lagrangian does have a good field expansion around the vacuum.

⁶In the same way that the polar angle is not a good coordinate around $r = 0$.

4 Giant magnons and their solitonic avatars

“Giant magnon” is the name given to a soliton of the reduced F/G model in the context of string theory. In particular, the examples of S^5 and $\mathbb{C}P^3$ are directly relevant to the AdS/CFT correspondence for $AdS_5 \times S^5$ [1] and $AdS_4 \times \mathbb{C}P^3$ [17], respectively. In this section, we review the known giant magnons in the context of S^n and $\mathbb{C}P^n$ and use them to illustrate some of the more important ideas discussed in the previous section in a more concrete way. In particular, we will discuss the relation between the giant magnons and their relativistic solitonic avatars in the associated SSSG equations.

The sphere S^n corresponds to the symmetric space $SO(n+1)/SO(n)$ while the complex symmetric space $\mathbb{C}P^n$ corresponds to $SU(n+1)/U(n)$. In both cases the associated involution is

$$\sigma_-(\mathcal{F}) = \theta \mathcal{F} \theta^{-1}, \tag{4.1}$$

where

$$\theta = \text{diag}(-1, 1, \dots, 1). \tag{4.2}$$

In appendix C we explain how to map the spaces S^n and $\mathbb{C}P^n$, expressed in terms of their usual coordinates, into the group field \mathcal{F} . For the spheres, parameterized by a real unit $n+1$ -vector \mathbf{X} with components X_a , $|\mathbf{X}| = 1$, we have

$$\mathcal{F} = \theta (1 - 2\mathbf{X}\mathbf{X}^T), \tag{4.3}$$

while for the complex projective spaces $\mathbb{C}P^n$ we have the complex $n+1$ vector \mathbf{Z} whose components are the complex projective coordinates Z_a , $a = 1, \dots, n+1$, so that $\mathbb{C}P^n$ is identified by modding out by complex re-scalings $Z_a \sim \lambda Z_a$, $\lambda \in \mathbb{C}$. In this case

$$\mathcal{F} = \theta \left(1 - 2 \frac{\mathbf{Z}\mathbf{Z}^\dagger}{|\mathbf{Z}|^2} \right). \tag{4.4}$$

The giant magnons can be thought of as excitations around a “vacuum” which is the simplest solution to the equations-of-motion and Pohlmeyer constraints, (2.7) and (3.1). The vacuum solution has $f_\pm = 1$ and

$$\mathcal{F}_0 = \exp [x_+ \Lambda_+ + x_- \Lambda_-], \tag{4.5}$$

where, up to overall conjugation and generalizing (3.56),

$$\Lambda_+ = \Lambda_- \equiv \Lambda = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{4.6}$$

This solution corresponds to the following solution for the sphere and projective coordinates,

$$\mathbf{X}_0 = \mathbf{Z}_0 = \mathbf{e}_1 \cos t - \mathbf{e}_2 \sin t. \tag{4.7}$$

Here, $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ are a set of orthonormal vectors in \mathbb{R}^{n+1} . The physical interpretation is clear: the string is collapsed to a point which traverses a great circle on S^n defined by the

plane spanned by e_1 and e_2 at the speed of light.⁷ The vacuum solution actually carries infinite sigma model charge and so the physically meaningful charge is actually the charge relative to the vacuum,

$$\Delta \mathcal{Q}_L = \int_{-\infty}^{\infty} dx (\partial_0 \mathcal{F} \mathcal{F}^{-1} - \partial_0 \mathcal{F}_0 \mathcal{F}_0^{-1}) . \quad (4.8)$$

In particular, the component of this charge along the Lie algebra element Λ , up to scaling, is identified with $\Delta - J$, the difference between the scaling dimension and R charge of the associated operator in the boundary CFT:

$$\Delta - J = \frac{\sqrt{\lambda}}{8\pi} \text{Tr} (\Lambda \Delta \mathcal{Q}_L) , \quad (4.9)$$

where λ is the 't Hooft coupling.

For the sphere case, we can express the conserved charge (4.8) directly in terms of X :

$$\mathcal{Q}_{L,ab} = \int_{-\infty}^{\infty} dx (\partial_0 X_a X_b - X_a \partial_0 X_b) . \quad (4.10)$$

In particular, $\Delta - J = \frac{\sqrt{\lambda}}{2\pi} \Delta \mathcal{Q}_{L,12}$. There is an analogous equation for the complex projective spaces in terms of the projective coordinates.

S^n giant magnons. The original giant magnon was described by Hofman and Maldacena [3]. It is a solution which takes values in the subspace $S^2 \subset S^n$ picked out by 3 mutually orthonormal vectors $\{e_1, e_2, \Omega\}$. The vectors e_1 and e_2 are already fixed by the choice of vacuum solution, however the direction of Ω , which describes an $S^{n-2} \subset S^n$, plays the rôle of an internal collective coordinate of the magnon. If (θ, ϕ) are polar coordinates on S^2 , then the solution written down by Hofman and Maldacena is

$$\begin{aligned} \cos \theta &= \frac{\sin \frac{p}{2}}{\cosh x'} , \\ \tan(\phi - t) &= \tan \frac{p}{2} \tanh x' . \end{aligned} \quad (4.11)$$

Here, and in the following, we define the Lorentz boosted coordinates t' and x' :

$$x' = x \cosh \vartheta - t \sinh \vartheta , \quad t' = t \cosh \vartheta - x \sinh \vartheta , \quad (4.12)$$

where ϑ is the rapidity ($v = \tanh \vartheta$). For the Hofman-Maldacena magnon,

$$\tanh \vartheta = \cos \frac{p}{2} . \quad (4.13)$$

Notice that the magnon is not relativistic in the sense that the moving solution *is not* the Lorentz boost of the stationary solution. The reason is that the Pohlmeyer constraints (3.1) are not Lorentz covariant (this is discussed in more detail in appendix B). In terms of the unit vector \mathbf{X} , we can write this solution as

$$\begin{aligned} \mathbf{X} &= \left[\sin t \sin \frac{p}{2} \tanh x' - \cos t \cos \frac{p}{2} \right] e_1 \\ &+ \left[\cos t \sin \frac{p}{2} \tanh x' + \sin t \cos \frac{p}{2} \right] e_2 + \sin \frac{p}{2} \text{sech } x' \Omega . \end{aligned} \quad (4.14)$$

⁷Notice that the plane is determined by the choice of representative Λ .

It has sigma model charge

$$\Delta Q_L = -4 \left| \sin \frac{p}{2} \right| \Lambda, \quad (4.15)$$

relative to the vacuum.

The solitonic avatar of the Hofman-Maldacena giant magnon in the reduced SSSG model has vanishing gauge fields $A_+^{(L)} = A_-^{(R)} = 0$, while the non-vanishing elements of γ are

$$\gamma = \left(\begin{array}{cc|c} -1 & 0 & \mathbf{0}^T \\ 0 & -\cos \theta(x) & \sin \theta(x) \mathbf{\Omega}^T \\ \mathbf{0} & \sin \theta(x) \mathbf{\Omega} & \mathbf{1} + (\cos \theta(x) - 1) \mathbf{\Omega} \mathbf{\Omega}^T \end{array} \right), \quad (4.16)$$

where we have highlighted the 2×2 subspace associated to \mathbf{e}_1 and \mathbf{e}_2 . In the above, $\theta(x)$ (not to be confused with the polar angle above or the rapidity) is the soliton solution to the sine-Gordon equation

$$\partial_\mu \partial^\mu \theta = -\sin \theta, \quad (4.17)$$

which can be written

$$\theta = 4 \tan^{-1}(e^x). \quad (4.18)$$

Since the sine-Gordon equation is relativistic, in the sense that the moving solution is the Lorentz boost of the static solution, it is sufficient to write the solution above in the soliton rest frame. This soliton has vanishing charges $Q_L = Q_R = 0$.⁸

The second kind of solution is Dorey's dyonic giant magnon [4]. The relation of Dorey's magnon to the Hofman-Maldacena magnon is analogous to the relation between the dyon and monopole solutions in gauge theories in $3+1$ dimensions. In the latter case, the dyon is obtained by allowing the charge angle, a collective coordinate taking values in S^1 , to move around the circle with constant velocity. The non-trivial aspect of this is that the angular motion has a back-reaction on the original monopole. One way to think of what is happening is in terms of Manton's picture of geodesic motion [28]. The charge angle is an internal collective coordinate of the monopole and the idea is that one can make a time-dependent solution by allowing the internal collective coordinates to be time dependent. For low velocities the motion is simply geodesic motion on the moduli space defined by a metric which is constructed from the inner-product of the zero modes associated to the collective coordinates. In the present setting, it is not clear whether Manton's analysis applies directly because the Hofman-Maldacena giant magnon is a time-dependent solution rather than a time-independent one like the monopole. We have seen that the Hofman-Maldacena giant magnon has an internal collective coordinate $\mathbf{\Omega}$ which parameterizes an S^{n-2} . Dorey's solution corresponds to allowing $\mathbf{\Omega}$ to move around a great circle (the geodesic) in S^{n-2} . We can describe this motion by picking out two orthonormal vectors $\mathbf{\Omega}^{(i)}$, orthogonal to \mathbf{e}_1 and \mathbf{e}_2 , and then take (in the magnon's rest frame)

$$\mathbf{\Omega}(t) = \cos(t \sin \alpha) \mathbf{\Omega}^{(1)} + \sin(t \sin \alpha) \mathbf{\Omega}^{(2)}. \quad (4.19)$$

⁸Since $H = \text{SO}(n-1)$ is semi-simple for $n \geq 4$, these charges provide examples of the conserved quantities $Q_{R/L}^{(\text{ss})}$ defined in the paragraph after (3.28).

The parameter α sets the angular velocity. This motion has a back-reaction on the original solution and we can write the complete moving solution as

$$\begin{aligned} \mathbf{X} = & \left(t \cos \frac{p}{2} + \sin t \sin \frac{p}{2} \tanh(x' \cos \alpha) \right) \mathbf{e}_1 \\ & + \left(\sin t \cos \frac{p}{2} + \cos t \sin \frac{p}{2} \tanh(x' \cos \alpha) \right) \mathbf{e}_2 + \sin \frac{p}{2} \operatorname{sech}(x' \cos \alpha) \boldsymbol{\Omega}(t'), \end{aligned} \quad (4.20)$$

The parameter α and the rapidity ϑ are determined by two parameters p and r via

$$\cot \alpha = \frac{2r}{1-r^2} \sin \frac{p}{2}, \quad \tanh \vartheta = \frac{2r}{1+r^2} \cos \frac{p}{2}. \quad (4.21)$$

The Hofman-Maldacena magnon corresponds to the limit $r \rightarrow 1$ (or $\alpha \rightarrow 0$).

Notice that the back-reaction of the angular motion is simply taken care of by the replacement $x' \rightarrow x' \cos \alpha$. The dyonic giant magnon carries charge

$$Q_L = -\frac{2(1+r^2)}{r} \left| \sin \frac{p}{2} \right| \Lambda - \frac{2(1-r^2)}{r} \left| \sin \frac{p}{2} \right| h, \quad (4.22)$$

relative to the vacuum, where h is the generator of $\text{SO}(n+1)$ corresponding to rotations in the plane picked out by $\boldsymbol{\Omega}^{(i)}$:

$$h = \boldsymbol{\Omega}^{(1)} \boldsymbol{\Omega}^{(2)T} - \boldsymbol{\Omega}^{(2)} \boldsymbol{\Omega}^{(1)T}. \quad (4.23)$$

In the reduced SSSG model, the dyonic magnon gives a soliton for which the gauge fields do not vanish:

$$A_+^{(L)} = -A_-^{(R)} = \frac{\cos^2 \alpha \sin \alpha}{\cos(2\alpha) - \cosh(2x \cos \alpha)} h. \quad (4.24)$$

In addition, the ‘‘missing’’ components defined in (3.18) are

$$A_+^{(R)} = -A_-^{(L)}, \quad A_-^{(L)} = -A_+^{(R)}. \quad (4.25)$$

which means that only the temporal components of the currents $J_{R/L}^\mu$ are non-vanishing. In addition, we have $A_\mu^{(L)} = -A_\mu^{(R)}$ which, in the case when H is abelian, corresponds to the condition for axial gauging (3.32) in the Lagrangian formulation. The group field (in the rest frame) generalizes (4.16) in an obvious way:

$$\gamma = \left(\begin{array}{cc|c} -1 & 0 & \mathbf{0}^T \\ 0 & -\cos \theta(x) & \sin \theta(x) \boldsymbol{\Omega}(t)^T \\ \hline \mathbf{0} & \sin \theta(x) \boldsymbol{\Omega}(t) & \mathbf{1} + (\cos \theta(x) - 1) \boldsymbol{\Omega}(t) \boldsymbol{\Omega}(t)^T \end{array} \right). \quad (4.26)$$

where

$$\cos \theta(x) = 1 - 2 \cos^2 \alpha \operatorname{sech}^2(x \cos \alpha), \quad (4.27)$$

which includes the effects of the back reaction of the geodesic motion. The solution carries charges (3.27)

$$Q_L = -Q_R = \int_{-\infty}^{\infty} dx \frac{\cos^2 \alpha \sin \alpha}{\cos(2\alpha) - \cosh(x \cos \alpha)} h = \left(\alpha - \frac{\pi}{2} \right) h. \quad (4.28)$$

In particular, notice that the non-vanishing combination $Q_L - Q_R$ is the unambiguously defined charge according to the discussion in section 3. Notice that for these solutions there is no contribution from the boundary terms in (3.27). In addition, when $\mathfrak{M} = S^n$, $n > 3$, the subgroup $H = \text{SO}(n - 1)$ is non-abelian. However, we can still define local conserved currents and associated charges because $\phi_{R/L}$ and gauge fields $A_\mu^{(R/L)}$ lie in an abelian subgroup $\text{SO}(2) \subset H$. The dyonic soliton has a mass

$$M = 4 \cos \alpha . \tag{4.29}$$

The Lagrangian interpretation of these dyons depends of the choice of $\epsilon_{R/L}$, which fixes the form of the group of gauge transformations. In the gauged WZW Lagrangian formulation with vector gauging, which can be achieved for any n , these dyons would carry non-vanishing topological charge Q^T . However, for the particular case of $\mathfrak{M} = S^3$, when $H = \text{SO}(2)$ is abelian, it is also possible to define an axially gauged WZW theory, in which case the dyons carry non-vanishing Noether charge Q^N corresponding to vector $\text{SO}(2)$ transformations. In both cases, Q^T and Q^N are related to $Q_{R/L}$ by means of (3.53).

$\mathbb{C}P^n$ giant magnons. Motivated by its application to the investigation of the AdS/CFT correspondence for $AdS_4 \times \mathbb{C}P^3$ [17], the $\mathbb{C}P^3$ case has been discussed in some detail in the literature [9–11, 18, 19]. The giant magnon solutions described so far are all obtained by embeddings of the Hofman-Maldacena giant magnon and Dorey’s dyonic magnon. The Hofman-Maldacena giant magnon can be embedded in $\mathbb{C}P^n$ in two distinct ways. Firstly, by taking $S^2 \simeq \mathbb{C}P^1 \subset \mathbb{C}P^n$ [9]. If $\theta(x, t)$ and $\phi(x, t)$ is the solution in terms of polar coordinates in (4.11), then the projective coordinates are

$$\mathbf{Z} = e^{i\phi(2x, 2t)/2} \sin(\theta(2x, 2t)/2) \mathbf{e}_1 + e^{-i\phi(2x, 2t)/2} \cos(\theta(2x, 2t)/2) \mathbf{e}_2 . \tag{4.30}$$

The scaling of the spacetime coordinates here is necessary in order to be consistent with the scaling of the Pohlmeyer constraints in (3.1). In addition, in order that the solution is oriented with respect to the choice of vacuum in (4.5), we have to rotate it with an element of $\text{SU}(2) \subset \text{SU}(n)$, $\mathbf{Z} \rightarrow U \mathbf{Z}$,

$$U = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|c} e^{i\pi/4} & e^{i\pi/4} & 0 \\ e^{3i\pi/4} & e^{-i\pi/4} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) . \tag{4.31}$$

Notice that this solution has no internal collective coordinates. The charge carried by the magnon is

$$\Delta Q_L = -2 \left| \sin \frac{p}{2} \right| \Lambda , \tag{4.32}$$

relative to the vacuum.

This magnon corresponds to a soliton solution of the SSSG equations of the form

$$\gamma = \left(\begin{array}{cc|c} e^{i\psi} & 0 & 0 \\ 0 & e^{-i\psi} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) , \tag{4.33}$$

with $A_+^{(L)} = A_-^{(R)} = 0$. The field ψ then satisfies the sine-Gordon equation

$$\partial^\mu \partial_\mu \psi = 2 \sin(2\psi) . \tag{4.34}$$

The giant magnon solution (4.30) corresponds to the sine-Gordon kink

$$\psi = 2 \tan^{-1}(e^{2x}) + \frac{\pi}{2} . \tag{4.35}$$

The solution carries $H_L^{(-)} \times H_R^{(+)}$ charge

$$Q_L = -Q_R = \frac{1}{2}(\psi(\infty) - \psi(-\infty)) = \frac{\pi}{2} . \tag{4.36}$$

The inequivalent embedding of the Hoffman-Maldacena giant magnon is via the subspace $\mathbb{R}P^2 \subset \mathbb{C}P^n$ [10]. The solution for \mathbf{Z} is exactly equal to \mathbf{X} in (4.14), however the vector $\mathbf{\Omega}$ can now be taken to be a complex vector with $|\mathbf{\Omega}| = 1$.⁹ Consequently the soliton has internal collective coordinates associated to an S^{2n-3} . The fact that the solution is valued in $\mathbb{R}P^2$ is because the solution (4.14) is itself valued in S^2 and this fixes the complex scaling freedom, $\mathbf{Z} \rightarrow \lambda \mathbf{Z}$, up to the discrete element $\mathbf{Z} \rightarrow -\mathbf{Z}$ and a further quotient by this gives $\mathbb{R}P^2$. The dyonic magnon can be embedded in an analogous way when $n \geq 3$ via the subspace $\mathbb{R}P^3 \subset \mathbb{C}P^n$ [11].

5 Magnons and solitons by dressing the vacuum

One way to construct the magnon/soliton solutions is to use an approach known as the dressing transformation [13] which is closely related to the Bäcklund transformation. For magnons in string theory this approach has been described in detail in [14, 15]. For the SSSG theories, and in particular their gauged WZW formulations, such an approach was described in [24, 29]. Schematically the transformation takes a known solution — for example a trivial kind of solution that we call the “vacuum” — and adds in a soliton.¹⁰ It is an important fact that the dressing transformation is consistent with the Pohlmeyer reduction in the sense that if the original solution satisfies the constraint (3.1) then so will the dressed solution. In fact we shall see that the dressing transformation constructs both the magnon and its SSSG soliton avatar at the same time without the need to map one into the other.

The dressing procedure has been described for general symmetric space sigma models in [22] and we shall draw heavily on results derived there. The procedure begins by identifying a “vacuum” solution. In the present context, our vacuum solution will be the simplest solution that satisfies the Pohlmeyer constraint (3.1). This identifies it as (4.5) which naturally satisfies the Pohlmeyer constraints (3.1) with $f_\pm = 1$. In appendix D, we show that the dressing transformation directly produces a soliton of the SSSG equations (3.13) with vanishing $A_+^{(L)} = A_-^{(R)} = 0$:

$$\partial_-(\gamma^{-1} \partial_+ \gamma) = \frac{1}{4} [\Lambda_+, \gamma^{-1} \Lambda_- \gamma] . \tag{5.1}$$

⁹In addition, we should replace $\mathbf{\Omega}^T$ by $\mathbf{\Omega}^\dagger$ in (4.16).

¹⁰In section 6, we will show that in the principal chiral model cases multiple soliton solutions are sometimes produced where the solitons are all mutually at rest.

In addition, we find that the conditions

$$\gamma^{-1}\partial_+\gamma\Big|_{\mathfrak{h}_+} = \partial_-\gamma\gamma^{-1}\Big|_{\mathfrak{h}_-} = 0 \tag{5.2}$$

are satisfied.¹¹ Notice that this means that the solitons automatically satisfy the equations of the gauged WZW model (3.40) for any choice of gauging.

The strategy of [22] begins by defining the symmetric space sigma model F/G in terms of initially the subgroup $F \subset \text{SL}(n, \mathbb{C})$ via one, or possibly more, involutions that we denote collectively as σ_+ . In order to pick out the coset $F/G \subset F$, the second part of the construction involves the extra involution σ_- whose explicit form for S^n and $\mathbb{C}P^n$ have been given in (4.1). The dressing transformation is then constructed in $\text{SL}(n, \mathbb{C})$ and the involutions give constraints that ensure that the transformation is restricted to the quotient F/G . In this work, we shall focus on three examples in order to be concrete:

- (i) $F/G = \text{SU}(n) \times \text{SU}(n)/\text{SU}(n)$. These are the principal chiral models and in this case it is more convenient to formulate the sigma model directly in terms of an $\text{SU}(n)$ -valued field $\mathcal{F}(x)$. In this case, there is only a single involution

$$\sigma_+(\mathcal{F}) = \mathcal{F}^{\dagger^{-1}} \tag{5.3}$$

required, and the involution σ_- is absent.

- (ii) The complex projective spaces $\mathbb{C}P^n = \text{SU}(n+1)/\text{U}(n)$. In this case, there are two involutions $\sigma_+(\mathcal{F}) = \mathcal{F}^{\dagger^{-1}}$ and $\sigma_-(\mathcal{F}) = \theta\mathcal{F}\theta$, the latter defined in (4.1).
- (iii) $S^n = \text{SO}(n+1)/\text{SO}(n)$. In this case, there are three involutions $\sigma_+^{(1)}(\mathcal{F}) = \mathcal{F}^{\dagger^{-1}}$, $\sigma_+^{(2)}(\mathcal{F}) = \mathcal{F}^*$ and $\sigma_-(\mathcal{F}) = \theta\mathcal{F}\theta$, the latter defined in (4.1).

Starting in $\text{SL}(n, \mathbb{C})$, the equations-of-motion for the sigma model have the zero curvature form (2.11) which are the integrability conditions for the associated linear system

$$\begin{aligned} \partial_+\Psi(x; \lambda) &= \frac{\partial_+\mathcal{F}\mathcal{F}^{-1}}{1+\lambda}\Psi(x; \lambda), \\ \partial_-\Psi(x; \lambda) &= \frac{\partial_-\mathcal{F}\mathcal{F}^{-1}}{1-\lambda}\Psi(x; \lambda). \end{aligned} \tag{5.4}$$

Notice that the group field is simply

$$\mathcal{F}(x) = \Psi(x; 0). \tag{5.5}$$

The dressing transformation involves constructing a new solution Ψ of the linear system of the form

$$\Psi(x; \lambda) = \chi(x; \lambda)\Psi_0(x; \lambda) \tag{5.6}$$

in terms of an old one Ψ_0 , which in our case corresponds to the vacuum solution in (4.5):

$$\Psi_0(x; \lambda) = \exp\left[\frac{x_+}{1+\lambda}\Lambda_+ + \frac{x_-}{1-\lambda}\Lambda_-\right], \tag{5.7}$$

¹¹This latter result makes use of the orthogonal decompositions (3.19), which hold in general for symmetric spaces of definite signature.

By picking out the residues of $\partial_{\pm}\Psi(\lambda)\Psi(\lambda)^{-1}$ at $\lambda = \mp 1$ (which come entirely from the terms where the derivatives hit $\Psi_0(x; \lambda)$) it follows that

$$\partial_{\pm}\mathcal{F}\mathcal{F}^{-1} = \chi(\mp 1)\Lambda_{\pm}\chi(\mp 1)^{-1}. \quad (5.8)$$

This is a key result because it means that the dressing transformation preserves the Pohlmeyer reduction and, in addition, we have

$$f_{\pm} = \chi(\mp 1)\Phi, \quad (5.9)$$

where Φ is a, as yet, unknown element that commutes with Λ_{\pm} :

$$[\Lambda_{\pm}, \Phi] = 0, \quad (5.10)$$

which will be chosen so that

$$\gamma = f_{-}^{-1}f_{+} = \Phi^{-1}\chi(+1)^{-1}\chi(-1)\Phi \quad (5.11)$$

is valued in $G \subset F$. This will then guarantee that the dressing transformation will give a soliton solution in the associated SSSG model. In addition, we will see that f_{\pm} satisfy (3.4). We will find below that

$$\Phi = \mathcal{F}_0^{1/2} = \exp\left[\frac{x_{+}\Lambda_{+}}{2} + \frac{x_{-}\Lambda_{-}}{2}\right]. \quad (5.12)$$

We now briefly review the construction of the dressing factor $\chi(\lambda)$ following [22]. The general form is

$$\chi(\lambda) = 1 + \sum_i \frac{Q_i}{\lambda - \lambda_i}, \quad \chi(\lambda)^{-1} = 1 + \sum_i \frac{R_i}{\lambda - \mu_i} \quad (5.13)$$

where the residues Q_i and R_i are matrices of the form

$$Q_i = \mathbf{X}_i \mathbf{F}_i^{\dagger}, \quad R_i = \mathbf{H}_i \mathbf{K}_i^{\dagger}, \quad (5.14)$$

for vectors $\mathbf{X}_i, \mathbf{F}_i, \mathbf{H}_i$ and \mathbf{K}_i .¹²

Taking the residues of $\chi(\lambda)\chi(\lambda)^{-1} = 1$ at $\lambda = \lambda_i$ and μ_i , gives, respectively,

$$Q_i + \frac{Q_i R_j}{\lambda_i - \mu_j} = 0, \quad R_i + \frac{Q_j R_i}{\mu_i - \lambda_j} = 0, \quad (5.15)$$

which can be used to solve for \mathbf{X}_i and \mathbf{K}_i :

$$\mathbf{X}_i \Gamma_{ij} = \mathbf{H}_j, \quad \mathbf{K}_i (\Gamma^{\dagger})_{ij} = -\mathbf{F}_j, \quad (5.16)$$

where the matrix

$$\Gamma_{ij} = \frac{\mathbf{F}_i^{\dagger} \mathbf{H}_j}{\lambda_i - \mu_j}. \quad (5.17)$$

¹²Notice that i is not the vector index but rather labels a set of vectors associated to the poles of $\chi^{\pm 1}(\lambda)$.

It follows from the linear system (5.4) that

$$\partial_{\pm} \mathcal{F} \mathcal{F}^{-1} = (1 \pm \lambda) \partial_{\pm} \chi \chi^{-1} + \chi \Lambda_{\pm} \chi^{-1} . \quad (5.18)$$

Since the left-hand side is independent of λ , the residues of the right-hand side at $\lambda = \lambda_i$ and μ_i must vanish,¹³ giving

$$\begin{aligned} (1 \pm \lambda_i) (\partial_{\pm} Q_i) \left(1 + \frac{R_j}{\lambda_i - \mu_j}\right) + Q_i \Lambda_{\pm} \left(1 + \frac{R_j}{\lambda_i - \mu_j}\right) &= 0, \\ -(1 \pm \mu_i) \left(1 + \frac{Q_j}{\mu_i - \lambda_j}\right) \partial_{\pm} R_i + \left(1 + \frac{Q_j}{\mu_i - \lambda_j}\right) \Lambda_{\pm} R_i &= 0, \end{aligned} \quad (5.19)$$

which are solved by

$$(1 \pm \lambda_i) \partial_{\pm} \mathbf{F}_i^{\dagger} = -\mathbf{F}_i^{\dagger} \Lambda_{\pm}, \quad (1 \pm \mu_i) \partial_{\pm} \mathbf{H}_i = \Lambda_{\pm} \mathbf{H}_i . \quad (5.20)$$

The solutions of these equations are

$$\mathbf{F}_i = (\Psi_0(\lambda_i)^{\dagger})^{-1} \boldsymbol{\varpi}_i, \quad \mathbf{H}_i = \Psi_0(\mu_i) \boldsymbol{\pi}_i, \quad (5.21)$$

for constant complex n -vectors $\boldsymbol{\varpi}_i$ and $\boldsymbol{\pi}_i$.

It can then be shown by tedious computation (re-produced in appendix D) that the generic solution that arises from the dressing procedure gives γ , as in (5.11) with Φ as in (5.12), i.e.

$$\gamma = \mathcal{F}_0^{-1/2} \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2} \quad (5.22)$$

satisfies the equation-of-motion (5.1). Using the explicit formulae for $\chi(\lambda)$ and its inverse, we find that the G -valued field of the reduction is

$$\gamma = 1 - \frac{2}{(1 - \mu_i)(1 + \lambda_j)} \mathcal{F}_0^{-1/2} \mathbf{H}_i (\Gamma^{-1})_{ij} \mathbf{F}_j^{\dagger} \mathcal{F}_0^{1/2} . \quad (5.23)$$

So we see that the data of the dressing transformation constructs both the sigma model magnon and the soliton in the SSSG. There are simple formulae for the charges $\mathcal{Q}_{L,R}$ of the dressed solution defined in (2.9), and for \mathcal{Q}_L the formula follows directly from (5.18): since the right-hand side is independent of λ we can evaluate it at $\lambda = \infty$, which gives

$$\partial_{\pm} \mathcal{F} \mathcal{F}^{-1} = \pm \partial_{\pm} \sum_i Q_i + \Lambda_{\pm} . \quad (5.24)$$

The final term here is precisely $\partial_{\pm} \mathcal{F}_0 \mathcal{F}_0^{-1}$ and so the meaningful quantity to calculate is the charge of the dressed solution relative to the vacuum solution, and it follows directly that

$$\Delta \mathcal{Q}_L = \sum_i Q_i \Big|_{x=\infty} - \sum_i Q_i \Big|_{x=-\infty} . \quad (5.25)$$

In order to calculate the charge \mathcal{Q}_R , we use the fact that

$$\mathcal{F}^{-1} = \mathcal{F} \Big|_{\lambda_i \rightarrow \lambda_i^{-1}, \mu_i \rightarrow \mu_i^{-1}, \Lambda_{\pm} \rightarrow -\Lambda_{\pm}} , \quad (5.26)$$

¹³In the following we will assume that $\lambda_i \neq \mu_j$ for any pair i, j . If the contrary is true then additional conditions must be imposed as we shall see later with an example.

which can be proved directly. Hence, it follows that the charge relative to the vacuum solution is

$$\Delta Q_R = -\Delta Q_L \Big|_{\lambda_i \rightarrow \lambda_i^{-1}, \mu_i \rightarrow \mu_i^{-1}, \Lambda_{\pm} \rightarrow -\Lambda_{\pm}}. \quad (5.27)$$

Up till now we have described the Bäcklund transformation for $SL(n, \mathbb{C})$. However, as mentioned above we have to impose involution conditions in order to describe a particular symmetric space. As described in [22] for each choice of symmetric space there are a set of involutions that must be imposed. First of all, there is an involution (or possibly more than one) σ_+ that picks out $F \subset SL(n, \mathbb{C})$:

$$F = \{ \mathcal{F} \in SL(n, \mathbb{C}) \mid \sigma_+(\mathcal{F}) = \mathcal{F} \}. \quad (5.28)$$

Then there is a further involution σ_- , that is detailed above for the explicit examples we have in mind, that picks out F/G parameterized by \mathcal{F} (as explained in appendix C):

$$F/G \simeq \{ \mathcal{F} \in F \mid \sigma_-(\mathcal{F}) = \mathcal{F}^{-1} \}. \quad (5.29)$$

Notice also that the quotient group is identified as

$$G = \{ \gamma \in F \mid \sigma_-(\gamma) = \gamma \}. \quad (5.30)$$

This allows us to prove that (5.22) is, as claimed, valued in $G \subset F$. Using $\sigma_-(\mathcal{F}_0^{\pm 1/2}) = \mathcal{F}_0^{\mp 1/2}$ and $\sigma_-(\chi(\pm 1)) = \mathcal{F}^{-1} \chi(\pm 1) \mathcal{F}_0$ gives

$$\sigma_-(\gamma) = \mathcal{F}_0^{1/2} \cdot \mathcal{F}_0^{-1} \chi(+1)^{-1} \cdot \mathcal{F}^{-1} \chi(-1) \mathcal{F}_0 \cdot \mathcal{F}_0^{-1/2} = \gamma. \quad (5.31)$$

The involutions are each of the following four types:

$$\begin{aligned} \sigma_1(\mathcal{F}) &= \theta \mathcal{F} \theta^{-1}, & \sigma_2(\mathcal{F}) &= \theta \mathcal{F}^* \theta^{-1}, \\ \sigma_3(\mathcal{F}) &= \theta (\mathcal{F}^T)^{-1} \theta^{-1}, & \sigma_4(\mathcal{F}) &= \theta \mathcal{F}^{\dagger -1} \theta^{-1}, \end{aligned} \quad (5.32)$$

where θ is either a symmetric, antisymmetric, hermitian or anti-hermitian matrix. The involutions (σ_1, σ_3) are holomorphic while (σ_2, σ_4) are anti-holomorphic.

The correct way to impose these conditions on $\Psi(x; \lambda)$ are

$$\begin{aligned} \Psi(\lambda) &= \sigma_+(\Psi(\tilde{\lambda})), \\ \Psi(1/\lambda) &= \mathcal{F} \sigma_-(\Psi(\tilde{\lambda})), \end{aligned} \quad (5.33)$$

where $\tilde{\lambda} = \lambda, \lambda^*$, if σ_{\pm} is holomorphic or anti-holomorphic, respectively. Notice that if we take $\lambda = 0$ and use $\mathcal{F} = \Psi(0)$ and $\Psi(\infty) = 1$, yields the correct conditions

$$\sigma_+(\mathcal{F}) = \mathcal{F}, \quad \sigma_-(\mathcal{F}) = \mathcal{F}^{-1}. \quad (5.34)$$

Furthermore it is easy to see that the vacuum solution (5.7) satisfies these conditions since $\Lambda_{\pm} \in \mathfrak{a} \subset \mathfrak{p}$.

Written in terms in terms of $\chi(\lambda)$ the conditions (5.33) become

$$\begin{aligned} \chi(\lambda) &= \sigma_+(\chi(\tilde{\lambda})), \\ \chi(1/\lambda) &= \mathcal{F} \sigma_-(\chi(\tilde{\lambda})) \mathcal{F}_0^{-1}, \end{aligned} \quad (5.35)$$

which means that the two sets of poles $\{\lambda_i\}$ and $\{\mu_i\}$ must be separately invariant under $\lambda \rightarrow \tilde{\lambda}$ (for σ_+), or $\lambda \rightarrow 1/\tilde{\lambda}$ (for σ_-), for σ_1 and σ_2 , and mapping into each other for σ_3 and σ_4 .

Rather than describe all the different cases, we specialize in this work to the three examples (i)-(iii). Notice that in all our examples the group F is compact and the elements $\Lambda_{\pm}^{\dagger} = -\Lambda_{\pm}$. Since we are principally interested in the application to string theory, there are some additional conditions on Λ_{\pm} . The vacuum solution \mathcal{F}_0 should be a t -dependent, but x -independent, solution. This immediately requires that $\Lambda_+ = \Lambda_-$. Notice that in cases (ii) and (iii), the symmetric space has rank 1 and so either $\Lambda_+ = \Lambda_- \equiv \Lambda$ or $\Lambda_+ = -\Lambda_- \equiv \Lambda$, for a fixed element Λ (up to conjugation). For case (i), the principal chiral model, there are more general models with $\Lambda_+ \neq \Lambda_-$ that will be discussed elsewhere.

The issue of relativistic invariance is quite subtle. With the choice $\Lambda_+ = \Lambda_-$, the vacuum solution is t -dependent. Clearly, if we boost this solution then it will no longer satisfy the Pohlmeyer constraints (3.1). This fact is then inherited by the dressed solution. As we shall argue, although the solution is localized in the sense, for example, that the density of its charges \mathcal{J}_0^L and \mathcal{J}_0^R , relative to the vacuum, is localized at a certain position in space moving with a certain velocity, the solutions with different velocities are not related by boosts. On the contrary, the solution in the reduced SSSG theory does respect Lorentz transformations in the sense that the solutions with different velocities are related by boosts (see appendix B).

Let us identify the velocity of the dressed solution. The dependence on x is via $\Psi_0(\xi)$, where ξ is one of the λ_i or μ_i . The localized nature of the soliton arises because when ξ has a imaginary part, $\Psi_0(\xi)$ has an exponential dependence on x . Assuming that Λ is anti-hermitian, the relevant dependence is

$$\exp \left[i \operatorname{Im} \left(\frac{x_+}{1+\xi} + \frac{x_-}{1-\xi} \right) \Lambda \right] = \exp \left[2i \operatorname{Im} \left(\frac{t - \xi x}{1 - \xi^2} \right) \Lambda \right] \quad (5.36)$$

and this leads to exponential fall-off of the energy/charge density away from the centre which is located at the solution of

$$\operatorname{Im} \left(\frac{t - \xi x}{1 - \xi^2} \right) = 0 . \quad (5.37)$$

The velocity of the soliton is therefore

$$v = \frac{\operatorname{Im} (1 - \xi^2)^{-1}}{\operatorname{Im} \xi (1 - \xi^2)^{-1}} = \frac{2r}{1 + r^2} \cos \frac{p}{2}, \quad (5.38)$$

where $\xi = r e^{ip/2}$. Roughly speaking, the dressed solution describes N solitons (for $i, j = 1, \dots, N$) and λ_i is a parameter that determines the velocity of the i^{th} soliton via (5.38) with $\xi = \lambda_i$. However, for the cases S^n and $\mathbb{C}P^n$, the additional constraints mean that the solution actually represents less than N independent solitons.

On the other hand, in the reduced SSSG model, the complete dependence on t and x is through the combination

$$\mathcal{F}_0^{-1/2} \Psi_0(\xi) = \exp \left[\left(\frac{(1 + \xi^2)t}{1 - \xi^2} - \frac{2\xi x}{1 - \xi^2} \right) \Lambda \right] . \quad (5.39)$$

In this case, the model does have relativistic invariance and the expression above can be written

$$\exp \left[\left(-t' \sin \alpha - ix' \cos \alpha \right) \Lambda \right], \quad (5.40)$$

where (t', x') are the boosted coordinates defined in (4.12) and the parameter α and the rapidity ϑ are determined by r and p as in (4.21). The angle α sets both the size and the internal angular velocity of the soliton. In the rest frame of the soliton, $p = \pi$ or, equivalently, $\xi = ri$.

6 The principal chiral models

As described in section 2, we can either think of these theories as symmetric space sigma models on $\mathfrak{M} = G \times G / G$ (so as a theory on $G \times G$ with an involution σ_- that exchanges the two G factors) or more directly as a sigma model defined on the Lie group G (and thus not needing a σ_- involution). We shall follow the second option and, hence, we formulate the theory in terms of a G -valued field \mathcal{F} defined as a subgroup of $SL(n, \mathbb{C})$ by the involution(s) σ_+ , and in this approach there is no involution σ_- .

In this work we will only consider the choice $G = SU(n)$, where there is a single involution

$$\sigma_+(\mathcal{F}) = \mathcal{F}^{\dagger -1} \quad (6.1)$$

that, in the classification of [22], is of type σ_4 with $\theta = \mathbf{I}$. Invariance under σ_+ requires

$$\Psi(\lambda) = \Psi(\lambda^*)^{\dagger -1}, \quad (6.2)$$

which is satisfied by imposing $\mu_i = \lambda_i^*$ that in turn implies that

$$\mathbf{H}_i = \mathbf{F}_i, \quad \mathbf{K}_i = \mathbf{X}_i \quad (6.3)$$

in (5.14). This means that $\pi_i = \varpi_i$ for each i and, moreover, that

$$\mathbf{X}_i = \mathbf{K}_i = \mathbf{F}_j (\Gamma^{-1})_{ji}, \quad \Gamma_{ij} = \frac{\mathbf{F}_i^{\dagger} \mathbf{F}_j}{\lambda_i - \lambda_j^*}, \quad (6.4)$$

and

$$\chi(\lambda) = 1 + \frac{\mathbf{F}_i (\Gamma^{-1})_{ij} \mathbf{F}_j^{\dagger}}{\lambda - \lambda_j}, \quad \chi(\lambda)^{-1} = 1 - \frac{\mathbf{F}_i (\Gamma^{-1})_{ij} \mathbf{F}_j^{\dagger}}{\lambda - \lambda_i^*}. \quad (6.5)$$

Then, using (5.5) and (5.6), the $SU(n)$ principal chiral model magnon is

$$\mathcal{F} = \chi(0) \mathcal{F}_0 = \mathcal{F}_0 - \frac{\mathbf{F}_i (\Gamma^{-1})_{ij} \mathbf{F}_j^{\dagger} \mathcal{F}_0}{\lambda_j}, \quad (6.6)$$

while, according to (5.23), its solitonic avatar in the associated SSSG theory reads

$$\gamma = 1 - \frac{2}{(1 - \lambda_i^*)(1 + \lambda_j)} \mathcal{F}_0^{-1/2} \mathbf{F}_i (\Gamma^{-1})_{ij} \mathbf{F}_j^{\dagger} \mathcal{F}_0^{1/2}, \quad (6.7)$$

where

$$F_i = \Psi_0(\lambda_i^*) \varpi_i . \tag{6.8}$$

In general, we will have to multiply (6.6) and (6.7) by constant phase factors in order to enforce $\det \mathcal{F} = 1$ and $\det \gamma = 1$, respectively.

The rank of the symmetric space $\mathfrak{M} = G \times G/G$ coincides with the rank of the Lie group G . Therefore, unless $G = \text{SU}(2)$, it gives rise to different Pohlmeyer reductions whose interpretation in the context of string theory is still to be understood. They are specified by two elements Λ_{\pm} of the Cartan subalgebra of \mathfrak{g} which, in the defining representation, are anti-hermitian diagonal matrices. In this work we will only consider the reductions corresponding to

$$\Lambda_+ = \Lambda_- = i \text{diag}(\zeta_a) = \Lambda \tag{6.9}$$

so that \mathcal{F}_0 only depends on t . Then, the vacuum solution of the associated linear system is

$$\Psi_0(\lambda) = \text{diag} \left(e^{\Theta_a(\lambda)} \right) , \tag{6.10}$$

where

$$\Theta_a(\lambda) = \frac{i\zeta_a x_+}{1+\lambda} + \frac{i\zeta_a x_-}{1-\lambda} = i\zeta_a \left(\frac{2t}{1-\lambda^2} - \frac{2\lambda x}{1-\lambda^2} \right) . \tag{6.11}$$

Furthermore, we will restrict ourselves to the cases with $\zeta_a \neq \zeta_b$ for $a \neq b$ so that $H^{(+)} = H^{(-)} = \text{U}(1)^{n-1}$, which correspond to the so-called (parity symmetric) homogeneous sine-Gordon models [20]. Then, without loss of generality, we can order the ζ_a according to ¹⁴

$$\zeta_1 > \zeta_2 > \dots \zeta_n . \tag{6.12}$$

More general reductions with $\Lambda_+ \neq \Lambda_-$ will be discussed elsewhere.

One-soliton solutions are obtained by considering a single pole in $\chi(\lambda)$, and so

$$\chi(\lambda) = 1 + \frac{\xi - \xi^* \mathbf{F} \mathbf{F}^\dagger}{\lambda - \xi \mathbf{F}^\dagger \mathbf{F}} , \tag{6.13}$$

where

$$\mathbf{F} = \Psi_0(\xi^*) \varpi \tag{6.14}$$

for a complex n -vector ϖ . In terms of components

$$F_a = e^{\Theta_a(\xi^*)} \varpi_a . \tag{6.15}$$

The complex n -vector ϖ represents a set of collective coordinates for the solitons. Since $\chi(\lambda)$ and, hence, the soliton solutions are explicitly invariant under complex re-scalings $\varpi \rightarrow \lambda \varpi$, with $\lambda \in \mathbf{C}$, these collective coordinates span a $\mathbb{C}P^{n-1}$. Notice that constant shifts of the solitons in space and time act on the collective coordinates via

$$\varpi \longrightarrow \exp \left[\frac{\delta x_+}{1 + \xi^*} \Lambda + \frac{\delta x_-}{1 - \xi^*} \Lambda \right] \varpi . \tag{6.16}$$

¹⁴For those who are familiar with the HSG theories, this is equivalent to taking $\Lambda_+ = \Lambda_-$ inside the principal Weyl chamber with respect to the standard choice of the basis of simple roots.

So some of the collective coordinates fix the position of the soliton in space and determine the temporal origin. The interpretation of the remaining “internal” collective coordinates will emerge when we analyze the solutions in more detail.

First of all, we think of the $SU(n)$ principal chiral model magnons. Using (6.6), the group-valued field \mathcal{F} is given by

$$\mathcal{F}_{ab} = e^{ip/n} \left(\delta_{ab} e^{\Theta_a(0)} - \frac{\xi - \xi^*}{\xi} \frac{e^{\Theta_a(\xi^*)} \varpi_a \varpi_b^* e^{-\Theta_b(\xi) + \Theta_b(0)}}{\sum_c |\varpi_c|^2 e^{\Theta_c(\xi^*) - \Theta_c(\xi)}} \right), \quad (6.17)$$

where we have multiplied \mathcal{F} by the phase $e^{ip/n}$ in order to enforce $\det \mathcal{F} = 1$. This magnon carries $SU(n)_L \times SU(n)_R$ charges $Q_{R/L}$ whose value relative to the vacuum solution can be calculated using (5.25) and (5.27). The result is $\Delta Q_{R/L} = \text{diag}(\Delta Q_{R/L}^a)$ with

$$\begin{aligned} \Delta Q_L^a &= -2i \left| r \sin \frac{p}{2} \right| (\delta_{a,\min} - \delta_{a,\max}) \\ \Delta Q_R^a &= -2i \left| r^{-1} \sin \frac{p}{2} \right| (\delta_{a,\min} - \delta_{a,\max}), \end{aligned} \quad (6.18)$$

where $\xi = r e^{ip/2}$. Next, we look at the solitonic avatar of the magnon (6.17) in the SSSG model which is provided by (6.7). It reads

$$\gamma_{ab} = e^{iC} \left(\delta_{ab} - \frac{2(\xi - \xi^*)}{(1 - \xi^*)(1 + \xi)} \cdot \frac{e^{\Theta_a(\xi^*) - \Theta_a(0)/2} \varpi_a \varpi_b^* e^{-\Theta_b(\xi) + \Theta_b(0)/2}}{\sum_c |\varpi_c|^2 e^{\Theta_c(\xi^*) - \Theta_c(\xi)}} \right), \quad (6.19)$$

where we have multiplied γ by the constant phase

$$e^{iC} = \left(\frac{1 - r^2 + 2ri \sin \frac{p}{2}}{1 - r^2 - 2ri \sin \frac{p}{2}} \right)^{1/n} \quad (6.20)$$

to enforce $\det \gamma = 1$. This field configuration satisfies the equations-of-motion (3.13) with $A_-^{(L)} = A_+^{(R)} = 0$. Then, the value of the unambiguously well-defined Lorentz invariant SSSG charge $Q_L - Q_R$ carried by this soliton is provided by

$$\gamma(+\infty) \gamma^{-1}(-\infty) = e^{Q_L - Q_R}, \quad (6.21)$$

which follows from (3.22), (3.27), and $\gamma_0^{\text{vac}} = 1$. The result is $Q_L - Q_R = \text{diag}(Q_L^a - Q_R^a)$ with

$$Q_L^a - Q_R^a = 2i \arctan \left(\frac{2|r|}{r^2 - 1} \right) (\delta_{a,\min} - \delta_{a,\max}). \quad (6.22)$$

Finally, the mass of this SSSG soliton can be calculated using (3.29), which leads to

$$M = \frac{4|r|}{(r^2 + 1)} (\zeta_{\min} - \zeta_{\max}). \quad (6.23)$$

Eqs. (6.22) and (6.23) show that all the non-trivial SSSG solutions are obtained with $r > 0$, and that charge conjugation corresponds to $r \rightarrow 1/r$. Moreover, eqs. (6.18), (6.22), and (6.23) unravel the rôle of the collective coordinates ϖ . This solution is actually a superposition of “max”–“min” basic solitons, all mutually at rest, which exhibits that,

with the special choice $\Lambda_+ = \Lambda_-$, there are no forces between them. The basic solitons are associated to the pairs $(a, a + 1)$, with only ϖ_a and ϖ_{a+1} non-vanishing. Those with a knowledge of root systems will appreciate that these basic solitons are naturally associated to the simple roots of $SU(n)$ and a particular $SU(2) \subset SU(n)$.¹⁵ Then, the rôle of the collective coordinates ϖ_a is to fix the relative space-time positions of the basic solitons.

Let us analyze the $SU(2)$ case in more detail, since $SU(2) \simeq S^3$ and, in any case, this describes the basic solitons of the $SU(n)$ theory. We take

$$\zeta_1 = -\zeta_2 = \frac{1}{2} \Rightarrow \Lambda = \frac{i}{2} \text{diag}(1, -1) . \quad (6.24)$$

Shifting x_{\pm} as in (6.16), and using the overall scaling symmetry, allows us to fix without-loss-of-generality $\varpi = (1, 1)$. As with more general solutions, $\xi = r e^{ip/2}$ determines the velocity of the soliton as well as the angular velocity of the internal motion, the former as in (5.38). The solution has the explicit form

$$\mathcal{F} = \begin{pmatrix} e^{it} \left(\cos \frac{p}{2} + i \sin \frac{p}{2} \tanh(x' \cos \alpha) \right) & -i \sin \frac{p}{2} e^{-it'} \sin \alpha \operatorname{sech}(x' \cos \alpha) \\ -i \sin \frac{p}{2} e^{+it'} \sin \alpha \operatorname{sech}(x' \cos \alpha) & e^{-it} \left(\cos \frac{p}{2} - i \sin \frac{p}{2} \tanh(x' \cos \alpha) \right) \end{pmatrix} , \quad (6.25)$$

where $x' = x \cosh \vartheta - t \sinh \vartheta$ and $t' = t \cosh \vartheta - x \sinh \vartheta$ are the boosted coordinates. The parameter α and the rapidity ϑ are determined by the two parameters p and r via (4.21). Notice, that the moving solution is *not* the boost of the solution at rest because of the $e^{\pm it}$ factors. Using (6.18), the $SU(2)_L \times SU(2)_R$ charges relative to the vacuum carried by this magnon can be written as

$$\Delta Q_L = -4|r \sin \sigma| \Lambda , \quad \Delta Q_R = -4|r^{-1} \sin \sigma| \Lambda . \quad (6.26)$$

It is not difficult to check that (6.25) corresponds to Dorey's dyonic magnon (4.20).¹⁶

Next we turn to the soliton avatar of (6.25). In the rest frame ($\xi = ri$, or $p = \pi$), it is

$$\gamma = \frac{1}{r^2 + 1} \begin{pmatrix} r^2 - 1 - 2ri \tanh(x \cos \alpha) & ire^{-2it} \sin \alpha \operatorname{sech}(x \cos \alpha) \\ 2ire^{+it} \sin \alpha \operatorname{sech}(x \cos \alpha) & r^2 - 1 + 2ri \tanh(x \cos \alpha) \end{pmatrix} , \quad (6.28)$$

along with $A_+^{(L)} = A_-^{(R)} = 0$. Using (4.21) with $p = \pi$, in this equation

$$\cos \alpha = \frac{2r}{1 + r^2} , \quad \sin \alpha = \frac{1 - r^2}{1 + r^2} . \quad (6.29)$$

Then, the charge and mass carried by this SSSG soliton can be written as

$$Q_L - Q_R = 4 \arctan \left(\frac{2|r|}{r^2 - 1} \right) \Lambda , \quad M = \frac{8|r|}{(r^2 + 1)} = 4 \left| \sin \left(\frac{1}{2} \operatorname{Tr}[\Lambda(Q_L - Q_R)] \right) \right| . \quad (6.30)$$

¹⁵The composite nature of these solutions was already noticed in [21] in the context of the homogeneous sine-Gordon theories.

¹⁶The explicit relationship reads

$$\mathbf{X} = -\operatorname{Re}(\mathcal{F}_{11})\mathbf{e}_1 + \operatorname{Im}(\mathcal{F}_{11})\mathbf{e}_2 - \operatorname{Im}(\mathcal{F}_{12})\mathbf{\Omega}^{(1)} - \operatorname{Re}(\mathcal{F}_{12})\mathbf{\Omega}^{(2)} . \quad (6.27)$$

Eq. (6.28) provides the well known one soliton solutions of the complex sine-Gordon equation [30]. Notice that the $r = 1$ soliton is static. It is the embedding of the usual sine-Gordon soliton in the reduced $SU(2)$ principal chiral model. For this configuration, the charge $Q_L - Q_R$ is uniquely defined only modulo $4\pi\Lambda$, a feature that played an important rôle in the construction of the CSG scattering matrix proposed in [31].

In [26], it was shown that this SSSG soliton saturates a Bogomol'nyi-type bound, which explains the explicit relationship between mass and charge shown in (6.30). If we choose axial gauging, then $Q_L - Q_R$ corresponds to a $U(1)$ Noether charge and these solutions provide two-dimensional examples of Q -balls, which has been recently exploited to investigate some aspects of the dynamics of that type of extended solutions in quantum field theories [32].

7 Complex projective space

In this case the target space of the sigma model is the symmetric space $SU(n+1)/U(n)$. As we have described in section 5, this is picked out from the universal construction in $SL(n, \mathbb{C})$ by two involutions; $\sigma_+(\mathcal{F}) = \mathcal{F}^{\dagger -1}$, along with σ_- in (4.1). Notice that σ_- is of type σ_1 in the list (5.32) and is consequently holomorphic. The vacuum solution is defined in (4.5).

Turning to the dressing transformation, invariance under σ_- requires that

$$\Psi(1/\lambda) = \mathcal{F}\theta\Psi(\lambda)\theta^{-1} \tag{7.1}$$

and this means that the poles $\{\lambda_i\}$ must come in pairs ($\lambda_i, \lambda_{i+1} = 1/\lambda_i$) and we can think of a single soliton as being a pair of the basic solitons of the $SU(n)$ principal chiral model. In addition, the fact that the poles come in pairs, requires associated conditions for each pair $i = 1, 3, \dots$:

$$\varpi_{i+1} = \theta\varpi_i, \tag{7.2}$$

which in turn means that

$$\mathbf{F}_{i+1} = \Psi_0(1/\lambda_i^*)\theta\varpi_i = \mathcal{F}_0\theta\Psi_0(\lambda_i^*)\varpi_i = \mathcal{F}_0\theta\mathbf{F}_i. \tag{7.3}$$

Let us consider in more detail the one soliton solution obtained from a single pair of poles $\{\xi, 1/\xi\}$. The dressing factor is

$$\chi(\lambda) = 1 + \frac{Q_1}{\lambda - \xi} + \frac{Q_2}{\lambda - 1/\xi}, \tag{7.4}$$

and the matrix Γ_{ij} has components

$$\begin{aligned} \Gamma_{11} &= \frac{\beta}{\xi - \xi^*}, & \Gamma_{12} &= \frac{\xi^*\gamma}{|\xi|^2 - 1}, \\ \Gamma_{21} &= -\frac{\xi\gamma}{|\xi|^2 - 1}, & \Gamma_{22} &= -\frac{|\xi|^2\beta}{\xi - \xi^*}, \end{aligned} \tag{7.5}$$

where we have defined the two real numbers

$$\beta = \mathbf{F}^\dagger\mathbf{F}, \quad \gamma = \mathbf{F}^\dagger\mathcal{F}_0\theta\mathbf{F}, \tag{7.6}$$

where $\mathbf{F} \equiv \mathbf{F}_1$. Therefore

$$\begin{aligned} Q_1 &= \frac{1}{\Delta} \left[-\frac{|\xi^2|\beta}{\xi - \xi^*} \mathbf{F}\mathbf{F}^\dagger + \frac{\xi\gamma}{|\xi|^2 - 1} \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \right], \\ Q_2 &= \frac{1}{\Delta} \left[\frac{\beta}{\xi - \xi^*} \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger - \frac{\xi^* \gamma}{|\xi|^2 - 1} \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger \right]. \end{aligned} \quad (7.7)$$

In the above, we have defined

$$\Delta = \det \Gamma = \frac{|\xi|^2 \gamma^2}{(|\xi|^2 - 1)^2} - \frac{|\xi|^2 \beta^2}{(\xi - \xi^*)^2}. \quad (7.8)$$

The solution depends on the complex vector $\varpi \equiv \varpi_1$ and the complex number ξ . In addition,

$$\chi(\lambda)^{-1} = 1 + \frac{R_1}{\lambda - \xi^*} + \frac{R_2}{\lambda - 1/\xi^*}, \quad (7.9)$$

where

$$\begin{aligned} R_1 &= \frac{1}{\Delta} \left[\frac{|\xi^2|\beta}{\xi - \xi^*} \mathbf{F}\mathbf{F}^\dagger + \frac{\xi^* \gamma}{|\xi|^2 - 1} \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0 \right], \\ R_2 &= \frac{1}{\Delta} \left[-\frac{\beta}{\xi - \xi^*} \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger - \frac{\xi \gamma}{|\xi|^2 - 1} \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \right]. \end{aligned} \quad (7.10)$$

The magnon solution is obtained from $\mathcal{F} = \chi(0)\mathcal{F}_0$. It corresponds to the projective coordinates¹⁷

$$\mathbf{Z} = (\tilde{\alpha} + \theta \mathbf{F}\mathbf{F}^\dagger \theta) \mathbf{Z}_0, \quad (7.11)$$

where

$$\tilde{\alpha} = -\frac{\xi\beta}{\xi - \xi^*} - \frac{\gamma}{|\xi|^2 - 1}. \quad (7.12)$$

The complex $n+1$ -vector ϖ represents a set of collective coordinates for the magnon. In fact, it is easy to see that only this vector up to complex re-scalings $\varpi \rightarrow \lambda\varpi$ lead to inequivalent solutions. By making shifts in x_\pm , as in (6.16), we can set always set, say, $\varpi_2 = 0$ and then use the scale symmetry to set $\varpi_1 = i$,¹⁸ so that

$$\varpi = i\mathbf{e}_1 + \mathbf{\Omega}, \quad \mathbf{\Omega} \cdot \mathbf{e}_1 = \mathbf{\Omega} \cdot \mathbf{e}_2 = 0, \quad (7.13)$$

where the constant vector $\mathbf{\Omega}$ is the internal collective coordinates of the magnon. The explicit solution is rather cumbersome to write down,

$$\mathbf{Z} = Z_1 \mathbf{e}_1 + Z_2 \mathbf{e}_2 + Z_3 \mathbf{\Omega}, \quad (7.14)$$

where

$$\begin{aligned} Z_1 &= \tilde{\alpha} \cos t + \cos \left(\frac{-2e^{ip/2}rx + 2e^{ipt}}{e^{ip} - r^2} \right) \cos \left(\frac{-2e^{ip/2}rx + (r^2 + 1)e^{ipt}}{-1 + e^{ip}r^2} \right) \\ Z_2 &= -\tilde{\alpha} \sin t - \sin \left(\frac{-2e^{ip/2}rx + 2e^{ipt}}{e^{ip} - r^2} \right) \cos \left(\frac{-2e^{ip/2}rx + (r^2 + 1)e^{ipt}}{-1 + e^{ip}r^2} \right) \\ Z_3 &= -\cos \left(\frac{-2e^{ip/2}rx + (r^2 + 1)e^{ipt}}{-1 + e^{ip}r^2} \right), \end{aligned} \quad (7.15)$$

¹⁷This is similar to the Euclidean space formulae in [33].

¹⁸The fact that we choose i here will make it simpler to relate the solution to the case $\mathfrak{M} = S^n$.

and

$$\begin{aligned} \tilde{\alpha} = & \frac{e^{ip}}{1 - e^{ip}} \left[|\mathbf{\Omega}|^2 + \cos \left(\frac{4ir \sin \frac{p}{2} (-(r^2 + 1)x + 2rt \cos \frac{p}{2})}{2r^2 \cos p - 1 - r^4} \right) \right] \\ & + \frac{1}{1 - r^2} \left[|\mathbf{\Omega}|^2 - \cos \left(\frac{2(r^2 - 1)(2rx \cos \frac{p}{2} - (1 + r^2)t)}{2r^2 \cos p - 1 - r^4} \right) \right] \end{aligned} \quad (7.16)$$

The magnon carries $SU(n)$ charge which can be extracted from (5.25). The computation is simplified by noticing that the off-diagonal elements in $Q_i = \mathbf{F}_i(\Gamma^{-1})_{ij}\mathbf{F}_j^\dagger$ (those with $j \neq i$) vanish as $x \rightarrow \pm\infty$ and so do not contribute to the charge. This is because as $x \rightarrow \pm\infty$, β , as defined in (7.6), diverges exponentially, while γ , also defined in (7.6), remains bounded. The remaining two contribution to the charge are then easily evaluated to give

$$\Delta Q_L = -2 \frac{1 + r^2}{r} \left| \sin \frac{p}{2} \right| |\Lambda|. \quad (7.17)$$

The magnon solution that we have constructed above is apparently singular when $|\xi| = 1$, i.e. $r = 1$ or $\alpha = 0$. However, a regular solution in this limit can be constructed by imposing the additional condition that

$$\gamma = F^\dagger \mathcal{F}_0 \theta F = \varpi^\dagger \theta \varpi = 0, \quad (7.18)$$

which can be written as a condition on the internal collective coordinates,¹⁹

$$|\mathbf{\Omega}| = 1. \quad (7.19)$$

In this case, the matrix Γ is diagonal and the dressing transformation has the simpler form:

$$\chi(\lambda) = 1 + \frac{\xi - \xi^* \mathbf{F}\mathbf{F}^\dagger}{\lambda - \xi} \frac{1}{\beta} - \frac{\xi - \xi^* \mathcal{F}_0 \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^\dagger}{\lambda - \xi^*} \frac{1}{\beta}, \quad (7.20)$$

The solution can also be obtained from (7.14) by setting $|\mathbf{\Omega}| = 1$ and taking the limit $r \rightarrow 1$. It is not difficult to see that up to a re-scaling by

$$- \frac{\cosh^2 x'}{\sin \frac{p}{2}}, \quad (7.21)$$

the solution is precisely an embedding of the Hofman-Maldacena magnon in (4.11). With reference to the discussion in section 4, it is the one associated to $\mathbb{R}P^2 \subset \mathbb{C}P^n$.

The solitonic avatar of the magnon (7.14) in the SSSG theory is²⁰

$$\begin{aligned} \gamma = & 1 + \frac{2}{\Delta} \left[\frac{|\xi^2| \beta}{(\xi - \xi^*)(1 - \xi^*)(1 + \xi)} \mathcal{F}_0^{-1/2} \mathbf{F}\mathbf{F}^\dagger \mathcal{F}_0^{1/2} \right. \\ & + \frac{\xi \gamma}{(|\xi|^2 - 1)(1 - 1/\xi^*)(1 + \xi)} \mathcal{F}_0^{1/2} \theta \mathbf{F}\mathbf{F}^\dagger \mathcal{F}_0^{1/2} \\ & + \frac{\xi^* \gamma}{(|\xi|^2 - 1)(1 - \xi^*)(1 + 1/\xi)} \mathcal{F}_0^{-1/2} \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^{-1/2} \\ & \left. - \frac{\beta}{(\xi - \xi^*)(1 - 1/\xi^*)(1 + 1/\xi)} \mathcal{F}_0^{1/2} \theta \mathbf{F}\mathbf{F}^\dagger \theta \mathcal{F}_0^{-1/2} \right]. \end{aligned} \quad (7.22)$$

¹⁹In addition, it is necessary that $\varpi_2 = 0$.

²⁰In general this solution has $\det \gamma = e^{iC}$, for a constant C and so in order that $\gamma \in G$ we should re-scale it by an appropriate compensating factor, as is done below in the explicit expressions.

In the rest frame, $p = \pi$, this solution has the explicit form

$$\gamma = \left(\begin{array}{cc|c} \gamma_{11} & 0 & \mathbf{0}^T \\ 0 & \gamma_{22} & \gamma_{23} \mathbf{\Omega}^\dagger \\ \mathbf{0} & \gamma_{32} \mathbf{\Omega} & \mathbf{1} + (\gamma_{33} - 1) \mathbf{\Omega} \mathbf{\Omega}^\dagger \end{array} \right), \quad (7.23)$$

where

$$\begin{aligned} \gamma_{11} &= e^{2i\eta} \frac{(r-i)^2 |\mathbf{\Omega}|^2 + 2ir \cos 2T + (r^2 - 1) \cosh 2X}{(r+i)^2 |\mathbf{\Omega}|^2 - 2ir \cos 2T + (r^2 - 1) \cosh 2X}, \\ \gamma_{22} &= e^{2i\eta} \frac{(r-i)^2 e^{-2i\eta} |\mathbf{\Omega}|^2 - 2ir \cos 2T + (r^2 - 1) \cosh 2X}{(r+i)^2 |\mathbf{\Omega}|^2 + 2ir \cos 2T + (r^2 - 1) \cosh 2X}, \\ \gamma_{33} &= e^{-4i\eta/3} \frac{(r+i)^2 e^{2i\eta} |\mathbf{\Omega}|^2 - 2ir \cos 2T + (r^2 - 1) \cosh 2X}{(r-i)^2 |\mathbf{\Omega}|^2 + 2ir \cos 2T + (r^2 - 1) \cosh 2X}, \\ \gamma_{23} &= -e^{-i\eta/3} \frac{8r \sin(T + iX)}{(r-i)^2 |\mathbf{\Omega}|^2 + 2ir \cos 2T + (r^2 - 1) \cosh 2X}, \\ \gamma_{32} &= -e^{-i\eta/3} \frac{8r \sin(T - iX)}{(r-i)^2 |\mathbf{\Omega}|^2 + 2ir \cos 2T + (r^2 - 1) \cosh 2X}, \end{aligned} \quad (7.24)$$

where $e^{i\eta} = (r+i)/(r-i)$, and where

$$T = \frac{r^2 - 1}{r^2 + 1} t, \quad X = \frac{2r}{r^2 + 1} x. \quad (7.25)$$

These solutions have vanishing SSSG charges $Q_L = Q_R = 0$. The mass of the solution can be computed using the expression for the energy in (3.29) and one finds

$$M = \frac{8r}{1+r^2} = 4 \cos \alpha. \quad (7.26)$$

Notice that it is more meaningful to write the result in terms of the parameter α defined in (4.21). The energy of the general moving solution (3.29) is

$$\mathcal{E} = \frac{8r}{1+r^2} \left| \sin \frac{p}{2} \right|. \quad (7.27)$$

The solution with $|\xi| = 1$, is obtained by first taking the limit $|\mathbf{\Omega}| \rightarrow 1$ and then $r \rightarrow 1$ (note these limits do not commute). In this limit, and in the soliton rest frame,

$$\gamma = \left(\begin{array}{cc|c} -1 & 0 & \mathbf{0}^T \\ 0 & -1 + 2 \operatorname{sech}^2(x) & 2 \tanh(x) \operatorname{sech}(x) \mathbf{\Omega}^\dagger \\ \mathbf{0} & 2 \tanh(x) \operatorname{sech}(x) \mathbf{\Omega} & \mathbf{1} - 2 \operatorname{sech}^2(x) \mathbf{\Omega} \mathbf{\Omega}^\dagger \end{array} \right), \quad (7.28)$$

which is a static solution.

8 The spheres

In this case the target space of the sigma model is the symmetric space $S^n \simeq \text{SO}(n+1)/\text{SO}(n)$, and the symmetric space is picked out by the three involutions

$$\sigma_+^{(1)}(\mathcal{F}) = \mathcal{F}^{\dagger^{-1}}, \quad \sigma_+^{(2)}(\mathcal{F}) = \mathcal{F}^*, \quad \sigma_-(\mathcal{F}) = \theta \mathcal{F} \theta^{-1}, \quad (8.1)$$

where θ is given in (4.2). Notice that σ_- is of type σ_1 in the list (5.32) and is consequently holomorphic. The Pohlmeyer reduction is defined by taking Λ_{\pm} as in (4.6). If we compare with the discussion of $\mathbb{C}P^n$ the only difference is the reality condition $\mathcal{F}^* = \mathcal{F}$.

The simplest magnon solution is obtained by considering the dressing transformation with a pair of poles ξ and $1/\xi$, where ξ is a phase. The constraints on the collective coordinates are (with $\varpi_1 = \varpi$)

$$\varpi_2 = \theta \varpi, \quad \varpi^* = \theta \varpi, \quad \varpi^\dagger \theta \varpi = 0. \tag{8.2}$$

These are precisely the same conditions on the magnon of the $\mathbb{C}P^n$ case with $r = 1$, with an additional reality condition. So just as in (7.13) we have

$$\varpi = i e_1 + \Omega, \tag{8.3}$$

where now Ω is a real unit vector orthogonal to e_1 and e_2 . Hence, the magnon has an internal collective coordinate taking values in S^{n-2} . This magnon is precisely the Hofman-Maldacena magnon (4.11). The soliton in the associated SSSG theory is precisely the $r = 1$ solution in the $\mathbb{C}P^n$ case (7.28) with the additional restriction that Ω is real.

9 Dyonic magnons/solitons

One characteristic feature of the magnon/soliton solutions that we have generated using the dressing transformation acting on the vacuum solution is that they carry a non-trivial moduli space of internal collective coordinates [3, 14].

Usually when solitons have internal collective coordinates one expects there are more general solutions for which the collective coordinates become time dependent. For a static soliton, the resulting motion is simply geodesic motion on the moduli space corresponding to a metric which is constructed from the inner products of the associated zero modes. When the moduli space arises from the action of a global symmetry then the metric will be invariant under the symmetry. The situation is familiar for BPS monopoles in gauge theories. In this case the monopoles carry an internal S^1 moduli space which can be thought of as the U(1) charge orientation of the monopole. A more general solution, the dyon, exists where the angle parameterizing the S^1 rotates with constant angular velocity. An important lesson for our present situation is that the dyon solution now carries electric charge as a consequence of the motion. Finding the dyon is not easy because the motion of the collective coordinate has a non-trivial back-reaction on the original solution.

In the present context, it is important to understand the action of the symmetries on the collective coordinates. First of all, recall that the sigma model with target space a symmetric space has a global F symmetry under which $\mathcal{F} \rightarrow U \mathcal{F} \sigma_-(U^{-1})$, $U \in F$. Once the Pohlmeyer reduction is performed, this symmetry corresponds to $f_{\pm} \rightarrow U f_{\pm}$, which leaves the SSSG field $\gamma = f_-^{-1} f_+$ invariant. Notice that the vacuum solution is invariant under the subgroup $H \subset G \subset F$.²¹ Hence, the transformations $U \in H$ on a magnon have

²¹We are assuming here that $H^{(\pm)}$, the subgroups of G that commute with Λ_{\pm} , are equal to H since in this paper we have $\Lambda_+ = \Lambda_-$.

a well defined action on the collective coordinates $\varpi \rightarrow U\varpi$, i.e. $\Omega \rightarrow U\Omega$ in the $\mathbb{C}P^n$ and S^n cases. On the other hand, the SSSG theory exhibits a global $H_L \times H_R$ symmetry that acts as $f_{\pm} \rightarrow f_{\pm} h_{\pm}^{-1}$ or, equivalently, $\gamma \rightarrow h_- \gamma h_+^{-1}$, where $h_{\pm} \in H$. In particular, the vector subgroup $\gamma \rightarrow U\gamma U^{-1}$ of transformations leaves the vacuum invariant and acts as a transformation on the soliton's collective coordinates in the same way as above: $\varpi \rightarrow U\varpi$. So the symmetry group H action on the collective coordinates can be interpreted in terms of a transformation of both the magnon's and soliton's collective coordinates where $H \subset F$ and $H \subset G$, respectively. This symmetry will play an important rôle in fixing the geometry on the moduli space of collective coordinates.

For example, for the cases $\mathfrak{M} = \mathbb{C}P^n$, the general magnon/soliton, (7.14) and (7.23), has an internal collective coordinate Ω which is a complex $n - 1$ vector (presented as a $n + 1$ -vector orthogonal to e_1 and e_2). For the particular solution with $r = 1$, we have the additional constraint $|\Omega| = 1$, so that the moduli space of collective coordinates is S^{2n-3} . In both cases there is a natural action of $H = U(n - 1)$ on the moduli space. However, in this case the symmetry is not large enough to completely fix the metric on the moduli space.

For the case with $\mathfrak{M} = S^n$ the soliton has a moduli space of collective coordinates equal to S^{n-2} parameterized by the real unit length $n - 1$ vector Ω (again presented as an $n + 1$ -vector orthogonal to e_1 and e_2) on which there is a natural action of $H = SO(n - 1)$. In this case, the symmetry fixes the metric on the moduli space (up to overall scaling). Dorey's solution is precisely the dyon associated to the Hofman-Maldacena magnon for the case $\mathfrak{M} = S^3 = SO(4)/SO(3)$. In this case Ω is a unit 2-vector in the subspace spanned by e_3 and e_4 . Allowing it to rotate with constant angular velocity,

$$\Omega(t) = \cos(t \sin \alpha) e_3 + \sin(t \sin \alpha) e_4, \tag{9.1}$$

leads to Dorey's dyonic magnon. However, in order to compute the complete back-reacted solution, it is more convenient to notice that there is another realization of the $S^3 = SO(4)/SO(3)$ model as the principal chiral model for $G = SU(2)$. The explicit map is

$$\mathcal{F} = \begin{pmatrix} X_1 + iX_2 & iX_3 + X_4 \\ iX_3 - X_4 & X_1 - iX_2 \end{pmatrix} \in SU(2). \tag{9.2}$$

In the $SU(2)$ formulation, the dyonic magnon is just the *ordinary* magnon solution which we described in section 6. In particular, we wrote (9.1) in such a way that the parameter α is the same as the one that appears as a parameter of the $SU(2)$ magnon.

The dyonic magnon gives a dyonic generalization of the SSSG soliton as we described in section 4 for the more general case with $\mathfrak{M} = S^n$, $n > 3$. In particular, the solution has non-trivial gauge fields $A_L^{(+)}$ and $A_R^{(-)}$, and, as we also explained in section 4, the dyonic solution can also be embedded in $\mathbb{C}P^n$, for $n \geq 3$ by using the maps $S^3 \rightarrow \mathbb{R}P^3 \rightarrow \mathbb{C}P^n$. However, because the symmetry $H = U(n - 1)$ is not large enough to fix the metric on S^{2n-3} there should exist another inequivalent class of dyonic solutions. In more detail, invariance under $U(n - 1)$ fixes the metric to be a linear combination

$$ds^2 = d\Omega^\dagger \cdot d\Omega + \xi (d\Omega^\dagger \cdot \Omega - \Omega^\dagger \cdot d\Omega)^2, \tag{9.3}$$

up to overall scaling. When $\xi = 0$ we have the usual spherically symmetric metric and in this case there are no new dyon solutions. However, when $\xi \neq 0$, the new class of dyon solutions are associated to geodesics of the form

$$\Omega(t) = e^{ht} \mathbf{p}, \tag{9.4}$$

where we can choose the overall orientation so that $\mathbf{p} = (1, 0, \dots, 0)$. The allowed algebra element h can be found by solving the geodesic equations for the metric (9.3). There are two classes of solution, firstly

$$h = i \begin{pmatrix} 0 & \mathbf{w}^\dagger \\ \mathbf{w} & 0 \end{pmatrix}, \tag{9.5}$$

where \mathbf{w} is a complex $n - 2$ vector. These give the embeddings of Dorey's dyon. The new class corresponds to

$$h = i \begin{pmatrix} v & \mathbf{w}^T \\ \mathbf{w} & (8\xi - 1)v\mathbf{w}\mathbf{w}^T/|\mathbf{w}|^2 \end{pmatrix}, \tag{9.6}$$

where v is a real number and \mathbf{w} is a real $n - 2$ vector. This new class includes the simple example

$$\Omega(t) = (e^{ivt}, 0, \dots, 0). \tag{9.7}$$

Such dyons will carry charge lying in the abelian subalgebra defined by h . For example in the case $\mathfrak{M} = \mathbb{C}P^2$ considered at the end of section 3, $\Omega = \Omega$ is just a complex 1-vector (or number) and only the new class of dyons with $\Omega(t) = e^{ivt}$ will exist. In terms of the Lagrangian formulation via axial gauging in (3.63), the dyon will correspond to a solution for which $\tilde{\psi} = vt$. Finding the back-reaction on the fields $\varphi(x)$ and $\theta(x)$ is a difficult challenge that we will not solve here.

10 Conclusions and outlook

In this work we have considered the interplay between the magnons in the sigma model describing string motion of certain symmetric spaces and the solitons of the related SSSG equations. A notable result is that the dressing procedure produces the magnon and soliton at the same time without the need to implement the complicated map between the two systems. We have also described how the dressing procedure in its current understanding cannot produce the more general dyonic magnon/soliton solutions which involve the non-trivial motion of the internal collective coordinates. It would be interesting to try to find a generalization of the dressing method which produces such dyonic solutions directly from the vacuum. In this work we have restricted ourselves to the simplest compact symmetric spaces and also to the simplest single magnon/soliton solutions: generalizations will be presented elsewhere.

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A Relation to the gauged sigma model approach

In this appendix we summarize the relationship between the approach described in [12] (see also [23]) and the formulation of the F/G symmetric space sigma model used in section 2 in terms of the principal chiral model for F . In [12], the F/G symmetric space sigma model is formulated with two fields $f \in F$ and $B_\mu \in \mathfrak{g}$ subject to the gauge symmetry

$$f \rightarrow fg^{-1}, \quad B_\mu \rightarrow g(B_\mu + \partial_\mu)g^{-1}, \quad g \in G. \quad (\text{A.1})$$

If the Lie group F is simple, the nonlinear sigma model is defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2\kappa} \text{Tr}(J_\mu J^\mu), \quad (\text{A.2})$$

where the current $J_\mu = f^{-1}\partial_\mu f - B_\mu \rightarrow gJ_\mu g^{-1}$ is covariant under gauge transformations.

The relationship between the two formulations relies on the fact that the solution space of the F/G sigma model can be realized as a subspace of the solution space of the F principal chiral model, which is a consequence of the following result due to Cartan [34]: The smooth mapping

$$\Phi : F/G \rightarrow F, \quad \text{with} \quad fG \mapsto \Phi(fG) = \sigma_-(f)f^{-1}, \quad (\text{A.3})$$

is a local diffeomorphism of F/G onto the closed totally geodesic submanifold $M = \{f \in F : \sigma_-(f) = f^{-1}\}$, where σ_- is the involution of F that fixes $G \subset F$ and gives rise to the canonical decomposition (2.2). Examples of this map can be found in appendix C. Taking (A.3) into account, the explicit connection between the two models was worked out in [34] making use of the gauge-invariant field

$$\mathcal{F} = \sigma_-(f)f^{-1} \quad (\text{A.4})$$

that trivially satisfies the constraint (2.4); namely, $\sigma_-(\mathcal{F}) = \mathcal{F}^{-1}$. Notice that in (A.2) the gauge fields B_μ are just Lagrangian multipliers whose equations-of-motion are $J_\mu|_{\mathfrak{g}} = 0$, which is equivalent to

$$B_\mu = f^{-1}\partial_\mu f|_{\mathfrak{g}} \quad \text{and} \quad J_\mu = f^{-1}\partial_\mu f|_{\mathfrak{p}}. \quad (\text{A.5})$$

Then, it is easy to check that

$$\mathcal{J}_\mu = \partial_\mu \mathcal{F} \mathcal{F}^{-1} = -2\sigma_-(f)J_\mu\sigma_-(f^{-1}), \quad (\text{A.6})$$

and the Lagrangian (A.2) becomes

$$\mathcal{L} = -\frac{1}{2\kappa} \text{Tr}(J_\mu J^\mu) = -\frac{1}{8\kappa} \text{Tr}(\mathcal{J}_\mu \mathcal{J}^\mu), \quad (\text{A.7})$$

which is the Lagrangian of the F principal chiral model subject to the constraint (2.4). Moreover, using the identity

$$D_\mu J_\nu = \partial_\mu J_\nu + [B_\mu, J_\nu] = -\frac{1}{2} \sigma_-(f^{-1}) \left(\partial_\mu \mathcal{J}_\nu - \frac{1}{2} [\mathcal{J}_\mu, \mathcal{J}_\nu] \right) \sigma_-(f), \quad (\text{A.8})$$

the equations-of-motion of the F/G symmetric space sigma model become

$$D_\pm J_\mp = 0 \Rightarrow \partial_\pm \mathcal{J}_\mp - \frac{1}{2} [\mathcal{J}_\pm, \mathcal{J}_\mp] = 0, \quad (\text{A.9})$$

which are just (2.12).

Now, taking (A.6) into account, the constraints that specify the Pohlmeyer reduction of the model in terms of constrained principal chiral model field \mathcal{F} can be imported directly from the eqs. (3.11) and (3.17) of [12]:

$$\partial_\pm \mathcal{F} \mathcal{F}^{-1} = -2\sigma_-(f) J_\pm \sigma_-(f^{-1}) = -2\sigma_-(f) \left(\bar{g}_\pm (\mu_\pm \Lambda_\pm) \bar{g}_\pm^{-1} \right) \sigma_-(f^{-1}), \quad (\text{A.10})$$

where $\bar{g}_\pm \in G$, $\Lambda_\pm \in \mathfrak{a}$, and \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} in (2.2). They correspond to (3.1) with $f_\pm = \sigma_-(f) \bar{g}_\pm \in F$ where, for simplicity and without loss of generality, we have fixed $\mu_\pm = -\frac{1}{2}$. One can think of these overall scales multipliers as having been absorbed into Λ_\pm . Moreover, since $\bar{g}_\pm \in G$, it is straightforward to check that $\sigma_-(f_\pm) = \mathcal{F}^{-1} f_\pm$, and that $\gamma = \bar{g}_-^{-1} \bar{g}_+ = f_-^{-1} f_+$ takes values in G , in agreement with (3.4) and (3.5), respectively.

B Integrability, conserved currents, energy-momentum tensor and Lorentz transformations

In order to uncover the integrability of the SSSG equations (3.13), it is useful to formulate them as the zero curvature condition

$$[\mathcal{L}_+, \mathcal{L}_-] = 0, \quad (\text{B.1})$$

with the components of the Lax operator \mathcal{L}_μ given by

$$\begin{aligned} \mathcal{L}_+ &= \partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(L)} \gamma - \frac{1}{2} z \Lambda_+ \equiv \mathcal{L}_+(x_\pm, \gamma, A_+^{(L)}; z), \\ \mathcal{L}_- &= \partial_- + A_-^{(R)} - \frac{1}{2} z^{-1} \gamma^{-1} \Lambda_- \gamma \equiv \mathcal{L}_-(x_\pm, \gamma, A_-^{(R)}; z). \end{aligned} \quad (\text{B.2})$$

In the above z , the *spectral parameter*, is an arbitrary auxiliary parameter whose introduction plays a key rôle in establishing the integrability of the theory. The zero curvature condition gives rise to an infinite number of conserved densities labeled by their spin. The ones corresponding to spin 1 and 2 provide the usual Noether currents and the components

of the stress-energy tensor, respectively. It is important to recall that the zero curvature condition is subject to the gauge symmetry transformations

$$\gamma \rightarrow h_- \gamma h_+^{-1}, \quad A_-^{(R)} \rightarrow h_+ (A_-^{(R)} + \partial_-) h_+^{-1}, \quad A_+^{(L)} \rightarrow h_- (A_+^{(L)} + \partial_+) h_-^{-1}. \quad (\text{B.3})$$

We can deduce the form of those conserved densities using the ‘‘Drinfeld-Sokolov procedure’’ [35]. In order to do that, we notice that, with the introduction of the spectral parameter, the Lax operator can be written in terms of the affine algebra

$$\mathfrak{f}^{(1)} = \sum_{k \in \mathbf{Z}} \left(z^{2k} \otimes \mathfrak{g} + z^{2k+1} \otimes \mathfrak{p} \right) = \bigoplus_{k \in \mathbf{Z}} \mathfrak{f}_k^{(1)} \quad (\text{B.4})$$

by means of

$$z\Lambda_+ \equiv z \otimes \Lambda_+ \in \mathfrak{f}_1^{(1)}, \quad z^{-1}\Lambda_- \equiv z^{-1} \otimes \Lambda_- \in \mathfrak{f}_{-1}^{(1)}, \quad A_{\pm}^{(L/R)} \equiv 1 \otimes A_{\pm}^{(L/R)} \in \mathfrak{f}_0^{(1)}. \quad (\text{B.5})$$

Moreover, γ takes values in G that is the group associated to the Lie algebra $\mathfrak{f}_0^{(1)}$. Next, we introduce $\Phi_{(+)} \in \exp(\mathfrak{f}_{<0}^{(1)})$, and solve

$$\Phi_{(+)} \left(\partial_+ + \gamma^{-1} \partial_+ \gamma + \gamma^{-1} A_+^{(L)} \gamma - \frac{1}{2} z \Lambda_+ \right) \Phi_{(+)}^{-1} = \partial_+ - \frac{1}{2} z \Lambda_+ + h^{(+)}, \quad (\text{B.6})$$

with

$$h^{(+)} = \sum_{k \leq 0} z^{-k} h_{-k}^{(+)} \in \text{Ker}(\text{Ad}(\Lambda_+)) \cup \mathfrak{f}_{\leq 0}^{(1)}. \quad (\text{B.7})$$

Correspondingly,

$$\Phi_{(+)} \left(\partial_- + A_-^{(R)} - \frac{1}{2} z^{-1} \gamma^{-1} \Lambda_- \gamma \right) \Phi_{(+)}^{-1} = \partial_- + I^{(+)}, \quad I^{(+)} \in \mathfrak{f}_{\leq 0}^{(1)}. \quad (\text{B.8})$$

Then, the zero curvature condition implies

$$\left[\partial_+ - \frac{1}{2} z \Lambda_+ + h^{(+)}, \partial_- + I^{(+)} \right] = 0, \quad (\text{B.9})$$

The components of $h^{(+)}$ and $I^{(+)}$ on $\text{Cent}(\text{Ker}(\text{Ad}(\Lambda_+)))$ provide an infinite set of local conserved densities, while the other components provide non-local conserved ones. A second set of conserved quantities can be constructed starting from

$$\gamma \left(\partial_- + A_-^{(R)} - \frac{1}{2} z^{-1} \gamma^{-1} \Lambda_- \gamma \right) \gamma^{-1} = \partial_- - \partial_- \gamma \gamma^{-1} + \gamma A_-^{(R)} \gamma^{-1} - \frac{1}{2} z^{-1} \Lambda_- \quad (\text{B.10})$$

instead of \mathcal{L}_+ .

The explicit expression of the densities of spin 1 and 2 can be found by writing

$$\Phi_{(+)} = \exp \left(\sum_{k \geq 1} z^{-k} y_{-k} \right), \quad z^{-k} y_{-k} \in \mathfrak{f}_{-k}^{(1)}, \quad (\text{B.11})$$

and looking at the first components of (B.6), which read

$$h_0^{(+)} - \frac{1}{2}[\Lambda_+, y_{-1}] = \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(L)}\gamma \equiv q \quad (\text{B.12a})$$

$$h_{-1}^{(+)} - \frac{1}{2}[\Lambda_+, y_{-2}] = -\partial_+y_{-1} + [y_{-1}, q] - \frac{1}{4}[y_{-1}, [y_{-1}, \Lambda_+]] \quad (\text{B.12b})$$

...

Using (3.19), eq. (B.12a) provides

$$y_{-1} \in \text{Im}(\text{Ad}(\Lambda_+)), \quad h_0^{(+)} = \mathbf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^{(L)}\gamma) = A_+^{(R)}, \quad (\text{B.13})$$

where we have also used (3.18). In turn, (B.8) gives

$$I_0^{(+)} = A_-^{(R)}. \quad (\text{B.14})$$

Therefore, the 0-grade component of (B.9) on $\text{Ker}(\text{Ad}(\Lambda_+))$ reads

$$[\partial_+ + A_+^{(R)}, \partial_- + A_-^{(R)}] = 0, \quad (\text{B.15})$$

which is one of the two equations in (3.17). The other is obtained in a similar way starting from (B.10) instead of \mathcal{L}_+ . The local and non-local conserved quantities provided by these equations and their interpretation are extensively discussed in section 3.

The components of the stress-energy tensor are found by looking at the components of $h_{-1}^{(+)}$ and $I_{-1}^{(+)}$ along Λ_+ . Using (B.12b),

$$\text{Tr}(\Lambda_+ h_{-1}^{(+)}) = \text{Tr}\left(\Lambda_+ \left([y_{-1}, q] - \frac{1}{4}[y_{-1}, [y_{-1}, \Lambda_+]]\right)\right) = -\text{Tr}\left((q - h_0^{(+)})^2\right) \equiv 2T_{++}. \quad (\text{B.16})$$

Correspondingly, (B.8) provides

$$\text{Tr}(\Lambda_+ I_{-1}^{(+)}) = -\frac{1}{2}\text{Tr}\left(\Lambda_+ \gamma^{-1} \Lambda_- \gamma\right) \equiv -2T_{-+}, \quad (\text{B.17})$$

and (B.9) leads to

$$\partial_+ T_{-+} + \partial_- T_{++} = 0. \quad (\text{B.18})$$

The component T_{--} is obtained in a similar fashion starting from (B.10) instead of \mathcal{L}_+ . Then, the complete set of components of the energy-momentum tensor can be written as

$$T_{++} = -\frac{1}{2}\text{Tr}\left((q - h_0^{(+)})^2\right) = -\frac{1}{2}\text{Tr}\left[(\partial_+\gamma\gamma^{-1} + A_+^{(L)})^2 - A_+^{(R)2}\right] \quad (\text{B.19a})$$

$$T_{--} = -\frac{1}{2}\text{Tr}\left[(\gamma^{-1}\partial_-\gamma - A_-^{(R)})^2 - A_-^{(L)2}\right] \quad (\text{B.19b})$$

$$T_{-+} = T_{-+} = +\frac{1}{4}\text{Tr}\left[\Lambda_+ \gamma^{-1} \Lambda_- \gamma\right], \quad (\text{B.19c})$$

and it can be easily checked that these expressions are gauge invariant.

The formulation in terms of the Lax operator \mathcal{L}_\pm is also useful to discuss the behaviour of the reduced equations under Lorentz transformations. The SSSG equations (3.13) are Lorentz invariant, which means that given a solution

$$\gamma = \gamma(x_+, x_-), \quad A_+^{(L)} = A_+^{(L)}(x_+, x_-), \quad A_-^{(R)} = A_-^{(R)}(x_+, x_-) \quad (\text{B.20})$$

we can generate a boosted one by simply

$$\begin{aligned} \gamma &\rightarrow \gamma_\lambda = \gamma(\lambda^{-1}x_+, \lambda x_-), \\ A_+^{(L)} &\rightarrow A_+^{(L)}{}_\lambda = \lambda^{-1}A_+^{(L)}(\lambda^{-1}x_+, \lambda x_-), \\ A_-^{(R)} &\rightarrow A_-^{(R)}{}_\lambda = \lambda^{+1}A_-^{(R)}(\lambda^{-1}x_+, \lambda x_-). \end{aligned} \quad (\text{B.21})$$

This is equivalent to saying that the zero-curvature condition is invariant under the transformations

$$x_\pm \rightarrow \lambda^{\pm 1}x_\pm, \quad \gamma \rightarrow \gamma, \quad A_+^{(L)} \rightarrow \lambda^{-1}A_+^{(L)}, \quad A_-^{(R)} \rightarrow \lambda^{+1}A_-^{(R)}. \quad (\text{B.22})$$

Correspondingly, the Lax operators (B.2) transform as

$$\begin{aligned} \mathcal{L}_+(x_\pm, \gamma, A_+^{(L)}; z) &\rightarrow \lambda^{-1}\mathcal{L}_+(x_\pm, \gamma, A_+^{(L)}; \lambda z), \\ \mathcal{L}_-(x_\pm, \gamma, A_-^{(R)}; z) &\rightarrow \lambda^{+1}\mathcal{L}_-(x_\pm, \gamma, A_-^{(R)}; \lambda z). \end{aligned} \quad (\text{B.23})$$

In other words, the Lorentz transformation (B.22) is equivalent to the re-scaling of the spectral parameter $z \rightarrow \lambda z$, and the zero-curvature condition is invariant because it does not depend on z . Then, in (B.6) the Lorentz transformation (B.23) induces the following transformation on the conserved densities:

$$h_{-j}^{(+)} \rightarrow \lambda^{-1-j}h_{-j}^{(+)} \quad (\text{B.24})$$

which, in particular, shows that $h_0^{(+)}$ is of spin 1 (currents) and, therefore, that the corresponding conserved charges are Lorentz invariant.

In contrast to the SSSG equations, the Pohlmeyer reduced sigma model is not Lorentz invariant, as a consequence of the constraints (3.1). However, we can use the formulation of the former in term of the Lax operators \mathcal{L}_\pm to deduce a formal expression for the action of Lorentz transformations on the solutions to the reduced sigma model equations-of-motion. Consider the solutions to the z -dependent auxiliary linear problem

$$\mathcal{L}_+(x_\pm, \gamma, A_+^{(L)}; z)\Upsilon^{-1}(z) = \mathcal{L}_-(x_\pm, \gamma, A_-^{(R)}; z^{-1})\Upsilon^{-1}(z) = 0 \quad (\text{B.25})$$

where $\Upsilon(z) \equiv \Upsilon(x_\pm, \gamma, A_+^{(L)}, A_-^{(R)}; z)$, whose integrability conditions are provided by the zero-curvature equation (B.1). As explained in [12], in the gauged sigma model approach the reduced sigma model configuration corresponding to a given SSSG solution $\{\gamma, A_+^{(L)}, A_-^{(R)}\}$ is specified by the solution to (B.25) for $z = 1$; namely, $f = \Upsilon(1)$. Then, (B.23) shows that under a Lorentz transformation $\Upsilon(z) \rightarrow \Upsilon(\lambda z)$, which induces the following transformation of the reduced sigma model configuration:

$$f = \Upsilon(1) \longrightarrow f_\lambda = \Upsilon(\lambda). \quad (\text{B.26})$$

C The spheres and complex projective spaces

In this appendix, we explain how to map the spaces S^n and $\mathbb{C}P^n$, expressed in terms of their usual coordinates, into the group valued field \mathcal{F} given by (A.4).

A generic $f \in \text{SO}(n+1)$ satisfies $ff^T = 1$ which is equivalent to $f_{ac}f_{bc} = \delta_{ab}$. Then,

$$\mathcal{F} = \sigma_-(f)f^{-1} = \theta f \theta f^T \tag{C.1}$$

which, in terms of components, reads

$$\mathcal{F}_{ab} = \theta_{ac} \left(\delta_{cb} - 2f_{c1}f_{b1} \right). \tag{C.2}$$

Now, for a symmetric space $\mathfrak{M} = F/G$, we have to use that $F = I_0(\mathfrak{M})$ is the identity component of the group of isometries of \mathfrak{M} , and that it acts transitively on $\mathfrak{M} = F/G$. This means that $\mathfrak{M} = F \cdot \mathbf{p}_0$ for an arbitrary point $\mathbf{p}_0 \in \mathfrak{M}$ and, moreover, that G is the isotropy group (or little group) of \mathbf{p}_0 . In our case, for $S^n = \text{SO}(n+1)/\text{SO}(n)$ we can take $\mathbf{p}_0 = (1, 0, \dots, 0)$, so that the point corresponding to f is

$$\mathbf{X} = f \cdot \mathbf{p}_0 \Rightarrow X_a = f_{a1} \tag{C.3}$$

Then, (C.2) becomes

$$\mathcal{F} = \theta \left(1 - 2\mathbf{X}\mathbf{X}^T \right), \tag{C.4}$$

which is the parameterization we are looking for in terms of the unit vector \mathbf{X} , $|\mathbf{X}| = 1$. Notice that the map $\mathbf{X} \rightarrow \mathcal{F}$, which provides a particular example of (A.3), is surjective but not injective.

A similar argument can be followed for the case of the complex projective spaces, in which case (C.4) is replaced by

$$\mathcal{F} = \theta \left(1 - 2 \frac{\mathbf{Z}\mathbf{Z}^\dagger}{|\mathbf{Z}|^2} \right), \tag{C.5}$$

where \mathbf{Z} is a vector whose components are the usual $n+1$ projective coordinates of $\mathbb{C}P^n$. In this case, the map $\mathbf{Z} \rightarrow \mathcal{F}$ is one-to-one.

D The SSSG equations-of-motion

In this appendix we prove that the dressing procedure produces solutions of the SSSG equations-of-motion (3.13) with vanishing gauge fields. To start with, using (5.4) along with (5.6) and (5.8) one quickly deduces

$$\partial_{\pm}\chi(\lambda)\chi(\lambda)^{-1} = \frac{\chi(\mp 1)\Lambda_{\pm}\chi(\mp 1)^{-1} - \chi(\lambda)\Lambda_{\pm}\chi(\lambda)^{-1}}{1 \pm \lambda} \tag{D.1}$$

from which it follows that

$$\partial_{\pm}\chi(\pm 1)\chi(\pm 1)^{-1} = \frac{1}{2} \left(\chi(\mp 1)\Lambda_{\pm}\chi(\mp 1)^{-1} - \chi(\pm 1)\Lambda_{\pm}\chi(\pm 1)^{-1} \right). \tag{D.2}$$

By writing $\gamma = \mathcal{F}_0^{-1/2} \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2}$, and using $\partial_{\pm} \mathcal{F}_0 = \Lambda_{\pm} \mathcal{F}_0$, we have

$$\begin{aligned} \gamma^{-1} \partial_+ \gamma &= -\frac{1}{2} \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \chi(+1) \Lambda_+ \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2} + \frac{1}{2} \Lambda_+ \\ &\quad - \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(+1) \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2} \\ &\quad + \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(-1) \mathcal{F}_0^{1/2}. \end{aligned} \quad (\text{D.3})$$

Using the upper-sign identity (D.2), one sees that the third term cancels the first two, to leave

$$\gamma^{-1} \partial_+ \gamma = \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(-1) \mathcal{F}_0^{1/2}. \quad (\text{D.4})$$

Then

$$\begin{aligned} \partial_- (\gamma^{-1} \partial_+ \gamma) &= -\frac{1}{2} \Lambda_- \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(-1) \mathcal{F}_0^{1/2} \\ &\quad + \frac{1}{2} \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(-1) \mathcal{F}_0^{1/2} \Lambda_- \\ &\quad + \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ (\partial_- \chi(-1) \chi(-1)^{-1}) \chi(-1) \mathcal{F}_0^{1/2} \end{aligned} \quad (\text{D.5})$$

Next, we use the lower-sign identity (D.2) to re-write the third term as

$$\begin{aligned} \partial_+ (\partial_- \chi(-1) \chi(-1)^{-1}) &= -\frac{1}{2} \partial_+ \chi(-1) \Lambda_- \chi(-1)^{-1} + \frac{1}{2} \chi(-1) \Lambda_- \chi(-1)^{-1} \partial_+ \chi(-1) \chi(-1)^{-1} \\ &\quad + \frac{1}{2} \partial_+ \chi(+1) \Lambda_- \chi(+1)^{-1} - \chi(+1) \Lambda_- \chi(+1)^{-1} \partial_+ \chi(+1) \chi(+1)^{-1}. \end{aligned} \quad (\text{D.6})$$

The first two terms cancel the first two terms in (D.5) to leave

$$\begin{aligned} \partial_- (\gamma^{-1} \partial_+ \gamma) &= \frac{1}{2} \mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_+ \chi(+1) \Lambda_- \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2} \\ &\quad - \frac{1}{2} \mathcal{F}_0^{-1/2} \chi(-1) \chi(+1) \Lambda_- \chi(+1)^{-1} \partial_+ \chi(+1) \chi(+1)^{-1} \chi(-1) \mathcal{F}_0^{1/2}. \end{aligned} \quad (\text{D.7})$$

Finally, we use the upper-sign identity in (D.2) again and the fact that $[\Lambda_+, \Lambda_-] = 0$, to end up with

$$\partial_- (\gamma^{-1} \partial_+ \gamma) = \frac{1}{4} [\Lambda_+, \gamma^{-1} \Lambda_- \gamma]. \quad (\text{D.8})$$

This is (3.13) with $A_+^{(L)} = A_-^{(R)} = 0$.

The next thing to prove is that the constraints (3.30) are satisfied. Taking the residue of the upper sign in (D.1) at $\lambda = -1$, gives

$$\partial_+ \chi(-1) \chi(-1)^{-1} = -\partial_{\lambda} \chi(-1) \Lambda_+ \chi(-1)^{-1} + \chi(-1) \Lambda_+ \chi(-1)^{-1} \partial_{\lambda} \chi(-1) \chi(-1)^{-1}. \quad (\text{D.9})$$

Substituting this in (D.4), gives

$$\gamma^{-1} \partial_+ \gamma = -[\mathcal{F}_0^{-1/2} \chi(-1)^{-1} \partial_{\lambda} \chi(-1) \mathcal{F}_0^{1/2}, \Lambda_+]. \quad (\text{D.10})$$

with a similar expression for $\partial_- \gamma \gamma^{-1}$. Hence, $\gamma^{-1} \partial_+ \gamma$ and $\partial_- \gamma \gamma^{-1}$ are in the image of the adjoint action of Λ_{\pm} . Then, provided that the orthogonal decompositions (3.19) hold, which is always true if the symmetric space has definite signature, the constraints (3.30) are satisfied.

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[SPIRES](#)].
- [2] K. Zarembo, *Semiclassical Bethe ansatz and AdS/CFT*, *Comptes Rendus Physique* **5** (2004) 1081 [[hep-th/0411191](#)] [[SPIRES](#)];
M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [[hep-th/0412188](#)] [[SPIRES](#)];
J.A. Minahan, *A brief introduction to the Bethe ansatz in $N=4$ super-Yang-Mills*, *J. Phys. A* **39** (2006) 12657 [[SPIRES](#)];
N. Gromov, V. Kazakov and P. Vieira, *Classical limit of quantum σ -models from Bethe ansatz*, *PoS(Solvay)005* [[hep-th/0703137](#)] [[SPIRES](#)];
N. Beisert, *The S-matrix of AdS/CFT and Yangian Symmetry*, *PoS(Solvay)002* [[arXiv:0704.0400](#)] [[SPIRES](#)];
G. Arutyunov and S. Frolov, *Foundations of the $AdS_5 \times S^5$ Superstring. Part I*, [arXiv:0901.4937](#) [[SPIRES](#)].
- [3] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys. A* **39** (2006) 13095 [[hep-th/0604135](#)] [[SPIRES](#)].
- [4] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys. A* **39** (2006) 13119 [[hep-th/0604175](#)] [[SPIRES](#)];
H.-Y. Chen, N. Dorey and K. Okamura, *Dyonic giant magnons*, *JHEP* **09** (2006) 024 [[hep-th/0605155](#)] [[SPIRES](#)].
- [5] K. Pohlmeyer, *Integrable Hamiltonian Systems and Interactions Through Quadratic Constraints*, *Commun. Math. Phys.* **46** (1976) 207 [[SPIRES](#)].
- [6] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, [hep-th/0311139](#) [[SPIRES](#)].
- [7] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[SPIRES](#)].
- [8] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)] [[SPIRES](#)].
- [9] D. Gaiotto, S. Giombi and X. Yin, *Spin Chains in $N = 6$ Superconformal Chern-Simons-Matter Theory*, [arXiv:0806.4589](#) [[SPIRES](#)].
- [10] G. Grignani, T. Harmark and M. Orselli, *The $SU(2) \times SU(2)$ sector in the string dual of $N = 6$ superconformal Chern-Simons theory*, *Nucl. Phys. B* **810** (2009) 115 [[arXiv:0806.4959](#)] [[SPIRES](#)].
- [11] M.C. Abbott and I. Aniceto, *Giant Magnons in $AdS_4 \times CP^3$: embeddings, Charges and a Hamiltonian*, [arXiv:0811.2423](#) [[SPIRES](#)].
- [12] J.L. Miramontes, *Pohlmeyer reduction revisited*, *JHEP* **10** (2008) 087 [[arXiv:0808.3365](#)] [[SPIRES](#)].
- [13] V.E. Zakharov and A.V. Mikhailov, *On the integrability of classical spinor models in two-dimensional space-time*, *Commun. Math. Phys.* **74** (1980) 21 [[SPIRES](#)]; *Relativistically Invariant Two-Dimensional Models in Field Theory Integrable by the Inverse Problem Technique. (In Russian)*, *Sov. Phys. JETP* **47** (1978) 1017 [*Zh. Eksp. Teor. Fiz.* **74** (1978) 1953] [[SPIRES](#)].

- [14] M. Spradlin and A. Volovich, *Dressing the giant magnon*, *JHEP* **10** (2006) 012 [[hep-th/0607009](#)] [[SPIRES](#)];
 A. Jevicki, C. Kalousios, M. Spradlin and A. Volovich, *Dressing the Giant Gluon*, *JHEP* **12** (2007) 047 [[arXiv:0708.0818](#)] [[SPIRES](#)];
 C. Kalousios, M. Spradlin and A. Volovich, *Dressing the giant magnon. II*, *JHEP* **03** (2007) 020 [[hep-th/0611033](#)] [[SPIRES](#)].
- [15] A. Jevicki, C. Kalousios, M. Spradlin and A. Volovich, *Dressing the Giant Gluon*, *JHEP* **12** (2007) 047 [[arXiv:0708.0818](#)] [[SPIRES](#)].
- [16] G.W. Gibbons and N.S. Manton, *Classical and Quantum Dynamics of BPS Monopoles*, *Nucl. Phys. B* **274** (1986) 183 [[SPIRES](#)].
- [17] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N=6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[SPIRES](#)];
 M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons Theories and AdS_4/CFT_3 Correspondence*, *JHEP* **09** (2008) 072 [[arXiv:0806.1519](#)] [[SPIRES](#)].
- [18] D. Astolfi, V.G.M. Puletti, G. Grignani, T. Harmark and M. Orselli, *Finite-size corrections in the $SU(2) \times SU(2)$ sector of type IIA string theory on $AdS_4 \times CP^3$* , *Nucl. Phys. B* **810** (2009) 150 [[arXiv:0807.1527](#)] [[SPIRES](#)].
- [19] G. Grignani, T. Harmark, M. Orselli and G.W. Semenoff, *Finite size Giant Magnons in the string dual of $N = 6$ superconformal Chern-Simons theory*, *JHEP* **12** (2008) 008 [[arXiv:0807.0205](#)] [[SPIRES](#)].
- [20] C.R. Fernandez-Pousa, M.V. Gallas, T.J. Hollowood and J.L. Miramontes, *The symmetric space and homogeneous sine-Gordon theories*, *Nucl. Phys. B* **484** (1997) 609 [[hep-th/9606032](#)] [[SPIRES](#)];
 J.L. Miramontes and C.R. Fernandez-Pousa, *Integrable quantum field theories with unstable particles*, *Phys. Lett. B* **472** (2000) 392 [[hep-th/9910218](#)] [[SPIRES](#)];
 P. Dorey and J.L. Miramontes, *Mass scales and crossover phenomena in the homogeneous sine-Gordon models*, *Nucl. Phys. B* **697** (2004) 405 [[hep-th/0405275](#)] [[SPIRES](#)].
- [21] C.R. Fernandez-Pousa and J.L. Miramontes, *Semi-classical spectrum of the homogeneous sine-Gordon theories*, *Nucl. Phys. B* **518** (1998) 745 [[hep-th/9706203](#)] [[SPIRES](#)].
- [22] J.P. Harnad, Y. Saint Aubin and S. Shnider, *Backlund transformations for nonlinear σ -models with values in riemannian symmetric spaces*, *Commun. Math. Phys.* **92** (1984) 329 [[SPIRES](#)].
- [23] H. Eichenherr and M. Forger, *Higher local conservation laws for nonlinear σ -models on symmetric spaces*, *Commun. Math. Phys.* **82** (1981) 227 [[SPIRES](#)].
- [24] I. Bakas, Q.-H. Park and H.-J. Shin, *Lagrangian Formulation of Symmetric Space sine-Gordon Models*, *Phys. Lett. B* **372** (1996) 45 [[hep-th/9512030](#)] [[SPIRES](#)].
- [25] M. Grigoriev and A.A. Tseytlin, *Pohlmeyer reduction of $AdS_5 \times S^5$ superstring σ -model*, *Nucl. Phys. B* **800** (2008) 450 [[arXiv:0711.0155](#)] [[SPIRES](#)].
- [26] J.L. Miramontes, *T-duality in massive integrable field theories: the homogeneous and complex sine-Gordon models*, *Nucl. Phys. B* **702** (2004) 419 [[hep-th/0408119](#)] [[SPIRES](#)].
- [27] J. Zinn-Justin, *Quantum field theory and critical phenomena*, *Int. Ser. Monogr. Phys.* **113** (2002) 1 [[SPIRES](#)].

- [28] N.S. Manton, *A Remark on the Scattering of BPS Monopoles*, *Phys. Lett. B* **110** (1982) 54 [SPIRES].
- [29] Q.-H. Park, *Deformed coset models from gauged WZW actions*, *Phys. Lett. B* **328** (1994) 329 [hep-th/9402038] [SPIRES].
- [30] F. Lund and T. Regge, *Unified Approach to Strings and Vortices with Soliton Solutions*, *Phys. Rev. D* **14** (1976) 1524 [SPIRES];
F. Lund, *Example of a Relativistic, Completely Integrable, Hamiltonian System*, *Phys. Rev. Lett.* **38** (1977) 1175 [SPIRES];
B.S. Getmanov, *New Lorentz Invariant Systems with Exact Multi-Soliton Solutions*, *Pisma Zh. Eksp. Teor. Fiz.* **25** (1977) 132 [SPIRES];
Q.-H. Park and H.J. Shin, *Duality in complex sine-Gordon theory*, *Phys. Lett. B* **359** (1995) 125 [hep-th/9506087] [SPIRES].
- [31] N. Dorey and T.J. Hollowood, *Quantum scattering of charged solitons in the complex sine-Gordon model*, *Nucl. Phys. B* **440** (1995) 215 [hep-th/9410140] [SPIRES].
- [32] P. Bowcock, D. Foster and P. Sutcliffe, *Q-balls, Integrability and Duality*, *J. Phys. A* **42** (2009) 085403 [arXiv:0809.3895] [SPIRES].
- [33] R. Sasaki, *explicit forms of Backlund transformations for the grassmannian and CP^{N-1} σ -models*, *J. Math. Phys.* **26** (1985) 1786 [SPIRES].
- [34] H. Eichenherr and M. Forger, *More about nonlinear σ -models on symmetric spaces*, *Nucl. Phys. B* **164** (1980) 528 [Erratum *ibid.* **B 282** (1987) 745] [SPIRES].
- [35] M.F. De Groot, T.J. Hollowood and J.L. Miramontes, *Generalized Drinfeld-Sokolov hierarchies*, *Commun. Math. Phys.* **145** (1992) 57 [SPIRES];
J.L. Miramontes, *Tau-functions generating the conservation laws for generalized integrable hierarchies of KdV and affine-Toda type*, *Nucl. Phys. B* **547** (1999) 623 [hep-th/9809052] [SPIRES].